RESEARCH ARTICLE

Mathematical Study on Prey-Predator Ecological System Considering Holling Type - II in a Polluted Environment

Nisha Rathaur [1], Yogesh Mishra and Raveendra Babu A.

Department of Mathematics, Prestige Institute of Management and Research, Gwalior 474020 India

Abstract: The study focuses on a mathematical study of a prey-predator ecological system incorporating Holling Type-II response and discussed the dynamics of the system with pollutants. The aim of the study is to decode the dynamic framework under the influence of toxicants. In the system it is considered that only the prey population is negatively effected by pollutants / toxins. Using stability requirements, all the possible equilibrium points of the system are discussed for local stability. It has been noted that when the pollutants / toxicant effect present, the system under consideration will survive but population reduces. Lastly numerical simulation is done to validate the analytical findings.

Keywords: Prey-Predator; Pollutants; Stability; Jacobian Matrix.

1. Introduction

Ecologists still have a difficult time solving the problems that arise from the presence of contaminants in ecosystems since they affect the biological populations in both terrestrial and aquatic ecosystems. The development rate and carrying capacity of biological organisms are generally slowed down by pollutants and toxicants. Preserving species variety and preventing extinction are the overarching objectives of ecologists and environmentalists in the face of ecological stress.Mathematical models have lately emerged as crucial tools and techniques for researching prey-predator food cycles and forecasting the survival or extinction of species [1, 2, 3].

Because of its widespread existence and importance, the dynamical study of prey populations and the animals that make up these populations have long been and will continue to be the core subjects in the discipline of ecology. Despite their apparent simplicity, the dynamical systems involved in the mathematical modelling of

[1]corresponding author

predator-prey dilemmas frequently lead to complex and difficult difficulties when they are thoroughly studied and analysed [4]. The foundations of modelling in ecosystem populations involve revealing the relevant prey and predator through mathematical models that take into account certain aspects of well system behaviour [5, 6].

Prey predator ecological systems are dynamically modelled, and this process is frequently evolving. An organised mathematical model of the prey-predator can go forward to a clear understanding of the viable path to the necessary changes.Additionally, several writers have employed mathematical models to comprehend the holling type II responses on predator-prey systems [7, 8]. While some authors have included the Allee effect in the prey growth function and examined the holling type II models [4, 7], the logistic equation is typically thought of as the prey's growth in these mathematical models of predator and prey that employ holling type II functional responses [9]. In the last few years, a lot of research has been done using mathematical models to examine the dynamic behaviour of tri-trophic level food chains [10, 11].

In the last decade, several research have examined the impact of pollutants on biological populations in contaminated environments, employing mathematical models [1, 2, 3, 12, 13, 14, 15]. In a study by the authors [16], the food chain of nonlinear dynamics of algae/phytoplankton toxin emission on the system is examined in relation to a marine example including three species and a food chain ecosystem of algae, zooplankton, and molluscs. Another marine example [10] is a tri-species food chain ecosystem made up of phytoplankton and zooplankton fish. Through study, the researchers have discovered that the toxin-producing phytoplankton lowers the grazing pressure on zooplankton species and that the dynamics of food-chain systems exhibit very little chaotic behavior [17, 18, 19].

The focus of this study is on a nonlinear mathematical model of ecological prey and predator populations in a polluted environment, with a functional response of Holling type 2. The significance of ecological system survival has been emphasized, and numerical evidence is demonstrated with the use of MATLAB.

2. The Mathematical Model

In this mathematical model, x(t) is the density of prey population, y(t) is the density of intermediate predator population, z(t) is the top predator population and c(t) is the concentration of pollutants / toxicant. r is the intrinsic growth of prey, k is the carrying capacity, a_1 and a_2 are the predation rates of prey and intermediate predator populations. e_1 and e_2 are the conversion rates. b_1 is the death rate of prey

by direct toxicant concentration on prey population. d_2 and d_3 are the death rates of intermediate and top predator populations respectively. Q_0 is the harmful pollutant or toxin which is external input into the environment. b_2 is washout rate and d_4 is the perish of population by toxic concentration.
In the mathematical model $u/(1+u)$, $(u = x \; or \; y)$, is the interactions of populations, considered by the functional response of Holling type-II. The prey predator three population model is described by a nonlinear differential equations:

$$\begin{aligned} \frac{dx}{dt} &= x\left(r(1-\frac{x}{k}) - \frac{a_1 y}{1+x} - b_1 c\right) \\ \frac{dy}{dt} &= y\left(\frac{e_1 x}{1+x} - \frac{a_2 z}{1+y} - d_2\right) \\ \frac{dz}{dt} &= z\left(\frac{e_2 y}{1+y} - d_3\right) \\ \frac{dc}{dt} &= Q_0 - b_2 c - d_4 x c \end{aligned} \tag{1}$$

with initials: $x(0) > 0, y(0) > 0, z(0) > 0, c(0) = f(0) \geq 0$.

3. Model Analysis

Now we will discuss the the equilibrium points of model (1). The equilibrium points of the model are obtained by considering $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = \frac{dc}{dt} = 0$.

3.1. **Equilibria of Model.** The model (1) has four positive equilibrium in x, y, z, c space namely, $\hat{E}_0(0,0,0,\hat{c})$, $\ddot{E}_1(\ddot{x},0,0,\ddot{c})$, $\tilde{E}_2(\tilde{x},\tilde{y},0,\tilde{c})$ and $\bar{E}_3(\bar{x},\bar{y},\bar{z},\bar{c})$. We prove the existence of $\hat{E}_0$, $\ddot{E}_1$, $\tilde{E}_2$ and $\bar{E}_3$ as follows:
$\hat{E}_0(0,0,0,\hat{c})$ **point:** The existence of $\hat{E}_0$ is obvious.
From the fourth equation of (1), we get $Q_0 - b_2\hat{c} = 0$ that is

$$\hat{c} = \frac{Q_0}{b_2} \tag{2}$$

$\ddot{E}_1(\ddot{x},0,0,\ddot{c})$ **point:** From the first equation of (1), we get $r(1-\frac{\ddot{x}}{k}) - b_1\ddot{c} = 0$

$$\ddot{x} = \frac{k}{r}(r - b_1\ddot{c}) \tag{3}$$

From the fourth equation of (1), we get $Q_0 - b_2\ddot{c} - d_4\ddot{x}\ddot{c} = 0$

$$A_1\ddot{c}^2 - A_2\ddot{c} + A_3 = 0 \tag{4}$$

where, $A_1 = kd_4b_1$, $A_2 = rkd_4 + rb_2$, $A_3 = rQ_0$. The equation (4) is positive under conditions.
$\tilde{E}_2(\tilde{x},\tilde{y},0,\tilde{c})$ **point:** From the first equation of (1), we get $(r(1-\frac{\tilde{x}}{k}) - \frac{a_1\tilde{y}}{1+\tilde{x}} - b_1\tilde{y} = 0$

$$\tilde{y} = A_4x^2 + A_5x + A_6 \tag{5}$$

where, $A_4 = -\frac{r}{ka_1}$, $A_5 = \frac{r}{a_1} - \frac{r}{ka_1} - \frac{b_1\tilde{y}}{a_1}$, $A_6 = \frac{r}{a_1} - \frac{b_1\tilde{y}}{a_1}$.
From the second equation of (1), we get $\frac{e_1\tilde{x}}{1+\tilde{x}} - d_2 = 0$, that is

$$\tilde{x} = \frac{d_2}{e_1 - d_2} \tag{6}$$

if $e_1 > d_2$. From the fourth equation of (1), we get $Q_0 - b_2\tilde{c} - d_4\tilde{x}\tilde{c} = 0$

$$\tilde{c} = \frac{Q_0}{b_2 + d_4\tilde{x}}. \tag{7}$$

$\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$ **point:** From the first equation of (1), we get $(r(1 - \frac{\bar{x}}{k}) - \frac{a_1\bar{y}}{1+\bar{x}} - b_1c = 0$

$$\bar{y} = A_4x^2 + A_5x + A_6 \tag{8}$$

where, $A_4 = -\frac{r}{ka_1}$, $A_5 = \frac{r}{a_1} - \frac{r}{ka_1} - \frac{b_1\bar{c}}{a_1}$, $A_6 = \frac{r}{a_1} - \frac{b_1\bar{c}}{a_1}$.
From the second equation of (1), we get $\frac{e_1\bar{x}}{1+\bar{x}} - \frac{a_2\bar{z}}{1+\bar{y}} - d_2 = 0$

$$\bar{z} = \frac{e_1}{a_2}(\frac{\bar{x}}{1+\bar{x}}) + \frac{e_1}{a_2}(\frac{\bar{x}\bar{y}}{1+\bar{x}}) - \frac{d_2\bar{y}}{a_2} - \frac{d_2}{a_2} \tag{9}$$

From the third equation of (1), we get $\frac{e_2\bar{y}}{1+\bar{y}} - d_3 = 0$

$$\bar{y} = \frac{d_3}{e_2 - d_3} \tag{10}$$

if $e_2 > d_3$. From the fourth equation of (1), we get $Q_0 - b_2\bar{c} - d_4\bar{x}\bar{c} = 0$

$$c = \frac{Q_0}{b_2 + d_4\bar{x}}. \tag{11}$$

3.2. **Stability of Model.** The stability of considered equilibriums are discussed from the linearlization results of model (1) around the equilibrium point. For linearizing, a Jacobian matrix of model (1) for the equilibrium point $E = (x, y, z, c)$:

$$J(f(x,y,z,c)) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial c} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial c} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial c} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial c} \end{bmatrix}$$

where $\frac{dx}{dt} = f_1(x, y, z, c)$, $\frac{dy}{dt} = f_2(x, y, z, c)$, $\frac{dz}{dt} = f_3(x, y, z, c)$ and $\frac{dc}{dt} = f_4(x, y, z, c)$.
The matrix components of J(f(x,y,z,c)) is $\frac{\partial f_1}{\partial x} = r - \frac{2rx}{k} - \frac{a_1y}{1+x^2+2x} - b_1c$, $\frac{\partial f_1}{\partial y} = -\frac{a_1x}{1+x}$, $\frac{\partial f_1}{\partial z} = 0$, $\frac{\partial f_1}{\partial c} = -b_1x$, $\frac{\partial f_2}{\partial x} = \frac{e_1y}{1+x^2+2x}$, $\frac{\partial f_2}{\partial y} = \frac{e_1x}{1+x} - \frac{a_2z}{1+y^2+2y} - d_2$, $\frac{\partial f_2}{\partial z} = -\frac{a_2y}{1+y}$, $\frac{\partial f_2}{\partial c} = 0$, $\frac{\partial f_3}{\partial x} = 0$, $\frac{\partial f_3}{\partial y} = \frac{e_2z}{1+y^2+2y}$, $\frac{\partial f_3}{\partial z} = \frac{e_2y}{1+y} - d_3$, $\frac{\partial f_3}{\partial c} = 0$, $\frac{\partial f_4}{\partial x} = -d_4c$, $\frac{\partial f_4}{\partial y} = 0$, $\frac{\partial f_4}{\partial z} = 0$, $\frac{\partial f_4}{\partial c} = -b_2 - d_4x$.

Theorem 1: If $\hat{c} > \frac{r}{b_1}$ then the equilibrium point $\hat{E}_0(0, 0, 0, \hat{c})$ is unstable.
Proof: The equilibrium point E_1 is substituted to the matrix element of j(f(x,y,z,c)),

obtained by the matrix j(f(E_1)) is

$$J(f(E_0)) = \begin{bmatrix} r - b_1\hat{c} & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \\ 0 & 0 & -d_3 & 0 \\ -d_4\hat{c} & 0 & 0 & -b_2 \end{bmatrix}$$

So, the characteristic equation for $J(f(E_1))$ is

$$(r - b_1\hat{c} - \lambda)(-d_2 - \lambda)(-d_3 - \lambda)(-b_2 - \lambda) = 0, \tag{12}$$

then the eigenvalues obtained are $\lambda_1 = r - b_1\hat{c}, \lambda_2 = -d_2, \lambda_3 = -d_3, \lambda_4 = -b_2$.

Theorem 2: If $\ddot{x} < \frac{d_2}{e_1 - d_2}$, $\ddot{x} > \frac{(r - b_1\ddot{c} - b_2)k}{2r + d_4k}$, then the equilibrium point $\ddot{E}_1(\ddot{x}, 0, 0, \ddot{c})$ is locally asymptotically stable.

Proof: The equilibrium point $\ddot{E}_1$ is substituted to the matrix element of J(f(x,y,z,c)), obtained by the matrix $J(f(E_1))$ that is

$$J(f(E_1)) = \begin{bmatrix} P & -\frac{a_1\ddot{x}}{1+\ddot{x}} & 0 & -b_1\ddot{x} \\ 0 & R & 0 & 0 \\ 0 & 0 & -d_3 & 0 \\ -d_4\ddot{c} & 0 & 0 & -b_2 - d_4\ddot{x} \end{bmatrix}$$

where $P = r - \frac{2r\ddot{x}}{k} - b_1\ddot{c}$, $R = \frac{e_1\ddot{x}}{1+\ddot{x}} - d_2$. So, the characteristic equation for $J(f(E_2))$ is

$$(R - \lambda)(-d_3 - \lambda)(\lambda^2 + M_1\lambda + M_2) = 0 \tag{13}$$

where $M_1 = -r + \frac{2r\ddot{x}}{k} + b_1\ddot{c} + b_2 + d_4\ddot{x}$, $M_2 = -rb_2 - rd_4\ddot{x} + \frac{2rb_2\ddot{x}}{k} + \frac{2rd_4\ddot{x}^2}{k} + b_1b_2\ddot{c} + b_1d_4\ddot{c}\ddot{x} - b_1d_4\ddot{x}\ddot{c}$.

Then, the eigenvalues obtained are $\lambda_1 = \frac{e_1\ddot{x}}{1+\ddot{x}} - d_2$,$\lambda_2 = -d_3$.

For the equilibrium point E_2 to be locally asymptotically stable, it must be $\lambda_1 < 0$, M_1>0, M_2>0 *i.e.*,

$$\ddot{x} < \frac{d_2}{e_1 - d_2} \tag{14}$$

$$\ddot{x} > \frac{(r - b_1\ddot{c} - b_2)k}{2r + d_4k} \tag{15}$$

Theorem 3: If $y < \frac{d_3}{e_2 - d_3}$, $N_1N_2 > N_3$, then the equilibrium point $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, \tilde{c})$ is locally asymptotically stable.

Proof: The equilibrium point E_2 is substituted to the matrix element of J(f(x,y,z)), obtained by the matrix $J(f(E_2))$ that is

$$J(f(E_2)) = \begin{bmatrix} P - b_1\tilde{c} & -\frac{a_1\tilde{x}}{1+\tilde{x}} & 0 & -b_1\tilde{x} \\ \frac{e_1\tilde{y}}{(1+\tilde{x})^2} & S & -\frac{a_2\tilde{y}}{1+\tilde{y}} & 0 \\ 0 & 0 & T & 0 \\ -d_4\tilde{c} & 0 & 0 & -b_2 - d_4\tilde{x} \end{bmatrix}$$

where $P = r - \frac{2r\tilde{x}}{k} - \frac{a_1\tilde{y}}{(1+\tilde{x})^2}$, $S = \frac{e_1\tilde{x}}{1+\tilde{x}} - d_2$, $T = \frac{e_2\tilde{y}}{1+\tilde{y}} - d_3$.
So, the characteristic equation for $J(f(E_2))$ is

$$(T-\lambda)(\lambda^3 + N_1\lambda^2 + N_2\lambda + N_3) = 0 \tag{16}$$

where $N_1 = -r + \frac{2r\tilde{x}}{k} + \frac{a_1\tilde{y}}{(1+\tilde{x})^2} + b_1\tilde{c} - \frac{e_1\tilde{x}}{1+\tilde{x}} + d_2 + b_2 + d_4\tilde{x}$,
$N_2 = -b_1d_4\tilde{x}\tilde{c} + \frac{re_1\tilde{x}}{1+\tilde{x}} - rd_2 - \frac{2re_1\tilde{x}^2}{k(1+\tilde{x})} + \frac{2rd_2\tilde{x}}{k} - \frac{a_1e_1\tilde{y}\tilde{x}}{(1+\tilde{x})^3} + \frac{a_1d_2\tilde{y}}{(1+\tilde{x})^2} - \frac{b_1e_1\tilde{x}\tilde{c}}{1+\tilde{x}} + b_1d_2\tilde{c} + \frac{a_1e_1\tilde{x}\tilde{y}}{(1+\tilde{x})^3} - ((b_2 + d_4\tilde{x})(r - \frac{2r\tilde{x}}{k} - \frac{a_1\tilde{y}}{(1+\tilde{x})^2} - b_1\tilde{c} - d_2 + \frac{e_1\tilde{x}}{1+\tilde{x}}))$,
$N_3 = -\frac{b_1d_4e_1\tilde{x}^2\tilde{c}}{1+\tilde{x}} - \frac{b_1d_4d_2\tilde{x}\tilde{c}}{1+\tilde{x}} + ((b_2 + d_4\tilde{x})(\frac{re_1\tilde{x}}{1+\tilde{x}} - rd_2 - \frac{2re_1\tilde{x}^2}{k(1+\tilde{x})} + \frac{2rd_2\tilde{x}}{k} - \frac{a_1e_1\tilde{y}\tilde{x}}{(1+\tilde{x})^3} + \frac{a_1d_2\tilde{y}}{(1+\tilde{x})^2} - \frac{b_1e_1\tilde{x}\tilde{c}}{1+\tilde{x}} + b_1d_2\tilde{c} + \frac{a_1e_1\tilde{x}\tilde{y}}{(1+\tilde{x})^3}))$,
then the eigenvalues obtained are $\lambda_1 = T$, $y < \frac{d_3}{e_2-d_3}$.
According to Routh-Hurwitz's criteria for the equilibrium point E_2 to be locally asymptotically stable, the following conditions must be satisfied: $N_1N_2 > N_3$, $N_1 > 0$ $i.e.$, $-r + \frac{2r\tilde{x}}{k} + \frac{a_1\tilde{y}}{(1+\tilde{x})^2} + b_1\tilde{c} - \frac{e_1\tilde{x}}{1+\tilde{x}} + d_2 - b_2 - d_4\tilde{x} > 0$,$N_2 > 0$ $i.e.$, $-b_1d_4\tilde{x}\tilde{c} + \frac{re_1\tilde{x}}{1+\tilde{x}} - rd_2 - \frac{2re_1\tilde{x}^2}{k(1+\tilde{x})} + \frac{2rd_2\tilde{x}}{k} - \frac{a_1e_1\tilde{y}\tilde{x}}{(1+\tilde{x})^3} + \frac{a_1d_2\tilde{y}}{(1+\tilde{x})^2} - \frac{b_1e_1\tilde{x}\tilde{c}}{1+\tilde{x}} + b_1d_2\tilde{c} + \frac{a_1e_1\tilde{x}\tilde{y}}{(1+\tilde{x})^3} - ((b_2 + d_4\tilde{x})(r - \frac{2r\tilde{x}}{k} - \frac{a_1\tilde{y}}{(1+\tilde{x})^2} - b_1\tilde{c} - d_2 + \frac{e_1\tilde{x}}{1+\tilde{x}})) > 0$, $N_3 > 0$ i.e., $-\frac{b_1d_4e_1\tilde{x}^2\tilde{c}}{1+\tilde{x}} - \frac{b_1d_4d_2\tilde{x}\tilde{c}}{1+\tilde{x}} + ((b_2 + d_4\tilde{x})(\frac{re_1\tilde{x}}{1+\tilde{x}} - rd_2 - \frac{2re_1\tilde{x}^2}{k(1+\tilde{x})} + \frac{2rd_2\tilde{x}}{k} - \frac{a_1e_1\tilde{y}\tilde{x}}{(1+\tilde{x})^3} + \frac{a_1d_2\tilde{y}}{(1+\tilde{x})^2} - \frac{b_1e_1\tilde{x}\tilde{c}}{1+\tilde{x}} + b_1d_2\tilde{c} + \frac{a_1e_1\tilde{x}\tilde{y}}{(1+\tilde{x})^3})) > 0$.

Theorem 4: The equilibrium point $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$ is locally asymptotically stable.
Proof: The equilibrium point E_3 is substituted to the matrix element of J(f(x,y,z)), obtained by the matrix $J(f(E_3))$ that is

$$J(f(E_3)) = \begin{bmatrix} m_1 & -\frac{a_1\bar{x}}{1+\bar{x}} & 0 & -b_1\bar{x} \\ \frac{e_1\bar{y}}{(1+\bar{x})^2} & m_2 & -\frac{a_2\bar{y}}{1+\bar{y}} & 0 \\ 0 & \frac{e_2\bar{z}}{(1+\bar{y})^2} & m_3 & 0 \\ -d_4\bar{c} & 0 & 0 & m_4 \end{bmatrix}$$

where $m_1 = r - \frac{2r\bar{x}}{k} - \frac{a_1\bar{y}}{(1+\bar{x})^2} - b_1\bar{c}$, $m_2 = \frac{e_1\bar{x}}{1+\bar{x}} - \frac{a_2\bar{z}}{(1+\bar{y})^2} - d_2$, $m_3 = \frac{e_2\bar{y}}{1+\bar{y}} - d_3$, $m_4 = -b_2 - d_4\bar{x}$.
So, the characteristic equation for $J(f(E_3))$ is

$$\lambda^4 + U_1\lambda^3 + U_2\lambda^2 + U_3\lambda + U_4 = 0 \tag{16}$$

where,
$U_1 = -(m_1 + m_4)$,
$U_2 = -d_4\bar{c}b_1\bar{x} + m_4m_1 + (m_1 + m_4)(m_2 + m_3) + m_2m_3 + \frac{a_2e_2\bar{y}\bar{z}}{(1+\bar{y})^3} + \frac{a_1e_1\bar{x}\bar{y}}{(1+\bar{x})^3}$,
$U_3 = b_1d_4\bar{x}\bar{c}(m_2+m_3) - m_1m_4(m_2+m_3) - (m_1+m_4)(m_2m_3 + \frac{a_2e_2\bar{y}\bar{z}}{(1+\bar{y})^3}) - (m_3+m_4)\frac{a_1e_1\bar{x}\bar{y}}{(1+\bar{x})^3}$,
$U_4 = -b_1d_4\bar{x}\bar{c}(m_2m_3 + \frac{a_2e_2\bar{y}^2}{(1+\bar{y})^3}) + m_1m_2m_3m_4 + (m_1m_4)\frac{a_2e_2\bar{y}\bar{z}}{(1+\bar{y})^3} + (m_3m_4)\frac{a_1e_1\bar{x}\bar{y}}{(1+\bar{x})^3}$.
According to Routh-Hurwitz's criteria for the equilibrium point E3 to be locally asymptotically stable, the following conditions must be satisfied

$$U_1U_2U_3 > U_3^2 + U_1^2U_4,$$

$U_1 > 0$ $i.e.,$ $-(m_1 + m_4) > 0$,
$U_2 > 0$ $i.e.,$ $-d_4\bar{c}b_1\bar{x} + m_4m_1 + (m_1 + m_4)(m_2 + m_3) + m_2m_3 + \frac{a_2e_2\bar{y}\bar{z}}{(1+\bar{y})^3} + \frac{a_1e_1\bar{x}\bar{y}}{(1+\bar{x})^3} > 0$,
$U_3 > 0$ $i.e.,$ $b_1d_4\bar{x}\bar{c}(m_2 + m_3) - m_1m_4(m_2 + m_3) - (m_1 + m_4)(m_2m_3 + \frac{a_2e_2\bar{y}\bar{z}}{(1+\bar{y})^3}) - (m_3 + m_4)\frac{a_1e_1\bar{x}\bar{y}}{(1+\bar{x})^3} > 0$,
$U_4 > 0$ $i.e.,$ $-b_1d_4\bar{x}\bar{c}(m_2m_3 + \frac{a_2e_2\bar{y}^2}{(1+\bar{y})^3}) + m_1m_2m_3m_4 + (m_1m_4)\frac{a_2e_2\bar{y}\bar{z}}{(1+\bar{y})^3} + (m_3m_4)\frac{a_1e_1\bar{x}\bar{y}}{(1+\bar{x})^3} > 0$.

It is difficult to interpret the results in ecological terms from the above complicated expressions, however, numerical examples are taken and graphs are plotted to illustrate the dynamical behavior's of the system.

4. Model Simulation

Numerical simulations are carried out using Matlab software. Model (1) includes numerical simulations for every equilibrium point in addition to simulations with different parameter adjustments. The primary goal of simulation is to verify analytical conclusions using numerical simulations and to examine the dynamic behavior of prey predator populations in the presence of toxicants and crowding. In the simulation, every figure symbolizes stability.

- For $\ddot{E}_1(\ddot{x}, 0, 0, \ddot{c})$ point, the following set of parameters have been selected: $r = 1.2; k = 1.199; a_1 = 0.123; b_1 = 0.75; e_1 = 0.008; a_2 = 0.9; d_2 = 0.095; e_2 = 0.024; d_3 = 0.09551; Q_0 = 0.123; b_2 = 0.123; d_4 = 0.1985;$ and observed the following values:

  $$\ddot{x} = 0.8916, \ddot{y} = 0.0000, \ddot{z} = 0.0000, \ddot{c} = 0.4100.$$

  It is showing that the $\ddot{E}_1(\ddot{x}, 0, 0, \ddot{c})$ is stable system (see Fig. 1), $i.e.$, when the toxicant concentration is available in the model the prey population is surviving. In this model, the intermediate ($\ddot{y}$) and top ($\ddot{z}$) predator populations are absent.
- For $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, \tilde{c})$ point, the following set of parameters have been selected: $r = 3.12; k = 0.2990; a_1 = 0.5; b_1 = 0.053; e_1 = 0.089; a_2 = 0.05; d_2 = 0.02; e_2 = 0.024; d_3 = 0.09; Q_0 = 0.3823; b_2 = 2.123; d_4 = 0.6985;$ and observed the following values:

  $$\tilde{x} = 0.2902, \tilde{y} = 0.2239, \tilde{z} = 0.0000, \tilde{c} = 0.1645$$

  It is showing that the $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, \tilde{c})$ is stable (see Fig.2, 3, 4), $i.e.$, when the toxicant concentration is available in the model the prey and intermediate predator populations are surviving.

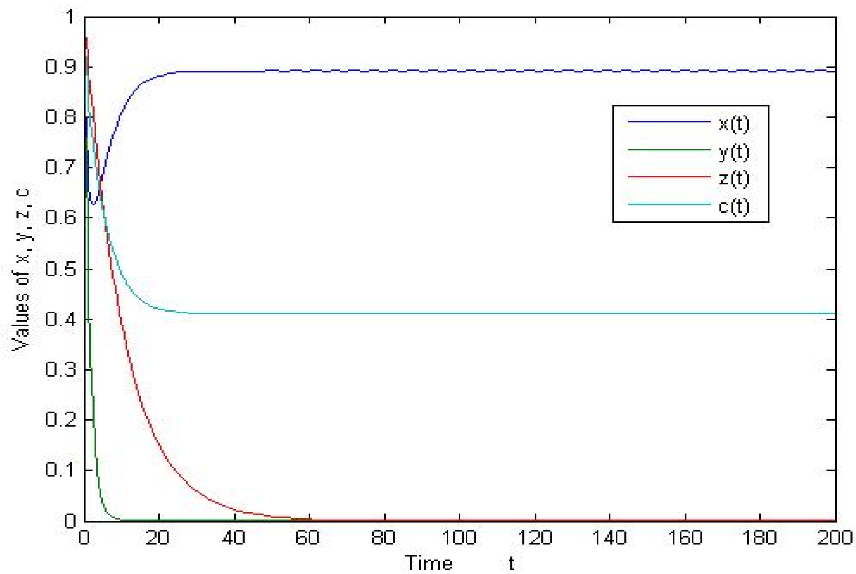

FIGURE 1. Stability behavior of (2.1) around the equilibrium point $\ddot{E}_1(\ddot{x}, 0, 0, \ddot{c})$.

- For $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$ point, the following set of parameters have been selected: $r = 3.12; k = 0.2990; a_1 = 0.5; b_1 = 0.053; e_1 = 0.089; a_2 = 0.05; d_2 = 0.02; e_2 = 0.024; d_3 = 0.09; Q_0 = 0.3823; b_2 = 2.123; d_4 = 0.6985;$ and observed the following values:

$$\bar{x} = 0.6051, \bar{y} = 0.2841, \bar{z} = 0.1191, \bar{c} = 0.0959$$

  It is showing that the $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$ is stable (see Fig.5, 6), *i.e.*, when the toxicant concentration is available in the model the prey population is surviving.
- For $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$ point, the following set of parameters have been selected: $r = 1.000001; k = 2.9956; a_1 = 3.9929975; d_1 = 1.973; a_2 = 0.008; b_1 = 0.9; c_1 = 0.095; b_2 = 0.249; c_2 = 0.024; Q_0 = 0.09551; d_2 = 0.1985; b = 0.223344;$ and observed that the system is unstable (see Fig.7, 8).

5. Conclusion

Considered a food chain mathematical which is based on the assumptions in the model, formulated with the effect of toxicant (see the system (1)). The considered prey predator mathematical model has four equilibriums, they are $\hat{E}_0(0, 0, 0, c)$, $\ddot{E}_1(\ddot{x}, 0, 0, c)$, $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, c)$ and $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, c)$. The survival of all the points have been found and also performed the stability analysis. The $\hat{E}_0(0, 0, 0, c)$ equilibrium point was obvious, other points $\ddot{E}_1(\ddot{x}, 0, 0, c)$, $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, c)$ and $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, c)$ were locally asymptotically stable under some analytical conditions. For numerical simulations we have used Matlab software. All the figures show the stability in nature. With the help of parameter values, all the analytical results have been verified. It has been

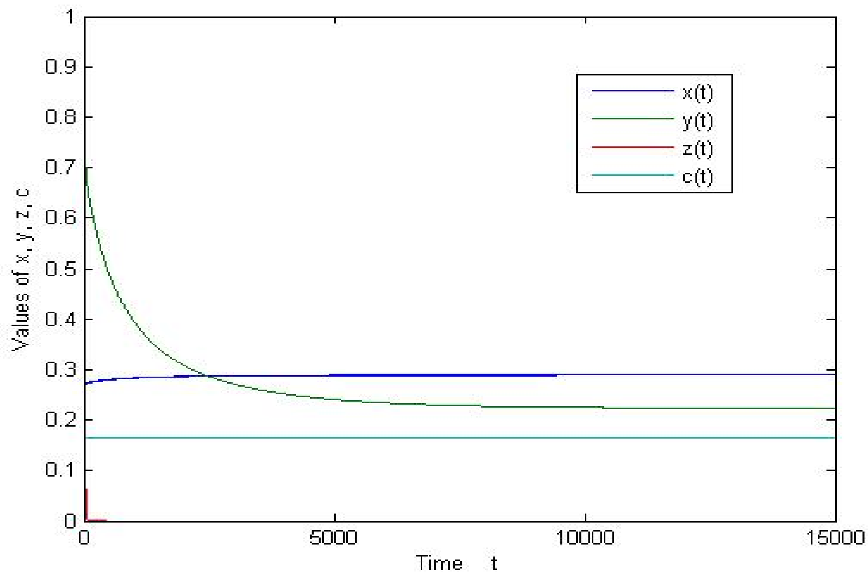

FIGURE 2. Stability behavior of (2.1) around the equilibrium point $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, \tilde{c})$.

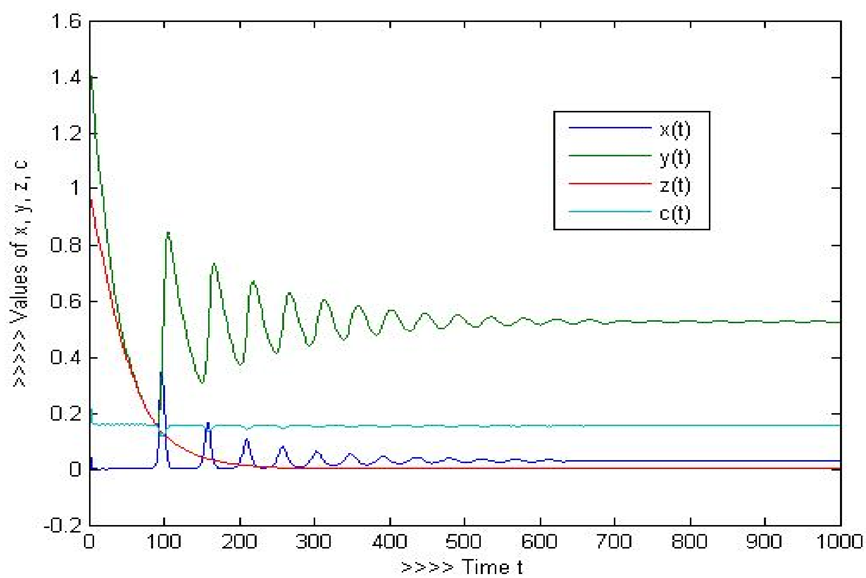

FIGURE 3. Stability behavior of (2.1) around the equilibrium point $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, \tilde{c})$.

observed that the system would be surviving less when the toxicant is present in the system.

References

[1] Turner, J. T. & Tester, P. A. (1997). Toxic marine phytoplankton, zooplankton grazers, and pelagic food webs, *Lim. Oce.*, **42** , 1203–1214. http://dx.doi.org/10.4319/lo.1997.42.5_part_2.1203 1

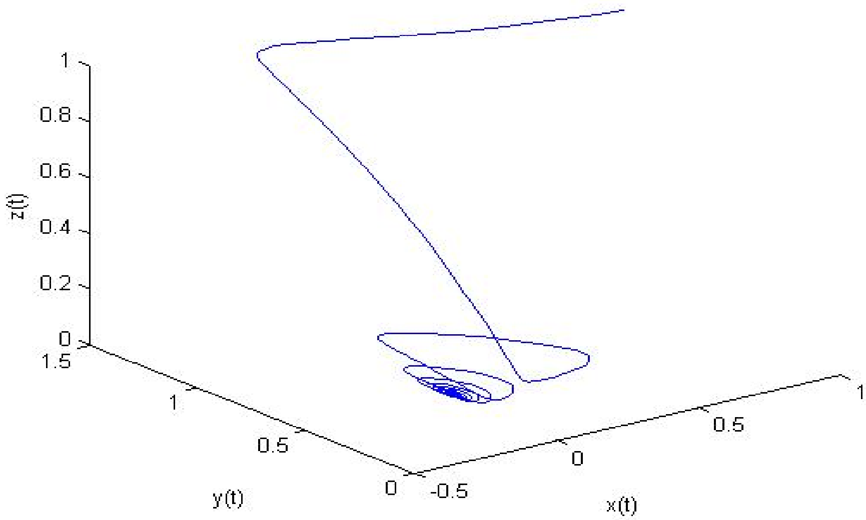

FIGURE 4. Phase diagram showing stability behavior of (2.1) around the equilibrium point $\tilde{E}_2(\tilde{x}, \tilde{y}, 0, \tilde{c})$.

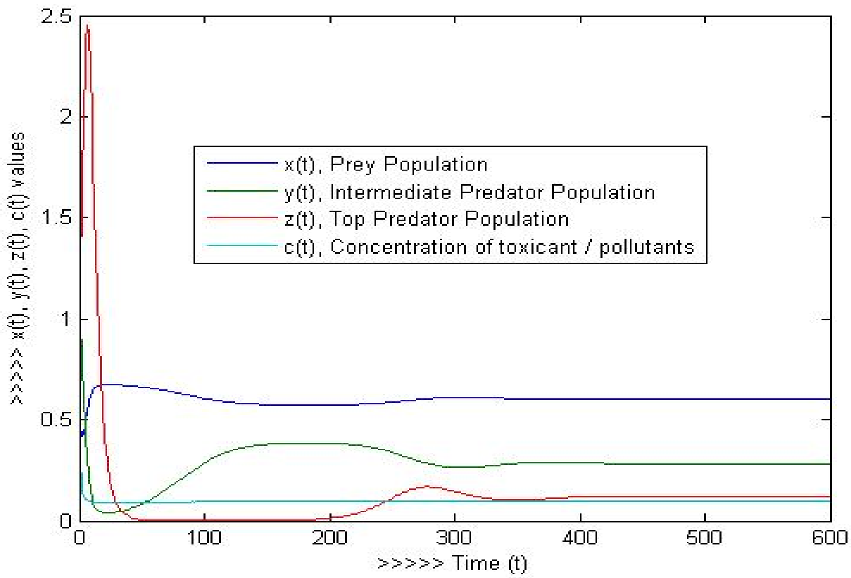

FIGURE 5. Stability behavior of (2.1) around the equilibrium point $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$.

[2] Babu A. R, & Sharma, M. (2022). The Impact of Toxicant on Prey Predator Ecological System: A Mathematical Approach, *Int. J. of Sci. Res. in Eng. and Mgmt.* (IJSREM) **6(8)**, 1-11. DOI: 10.55041/IJSREM16052 1

[3] Misra, O. P. & Babu A. R., (2016). Mathematical Study of a Leslie-Gower-type Tri- trophic Population Model in a Polluted Environment, *Model. Earth Syst. Environ.*, **2:29**, 1-11. https://link.springer.com/article/10.1007/s40808-016-0084-z 1

[4] Molla, S.H., Sarwardi, S. R., Smith & Haque, M., (2022). Dynamics of adding variable prey refuge and an Allee effect to a predator–prey model, *Alex. Eng. J.* **61(6)**, 4175-4188. https://doi.org/10.1016/j.aej.2021.09.039 1

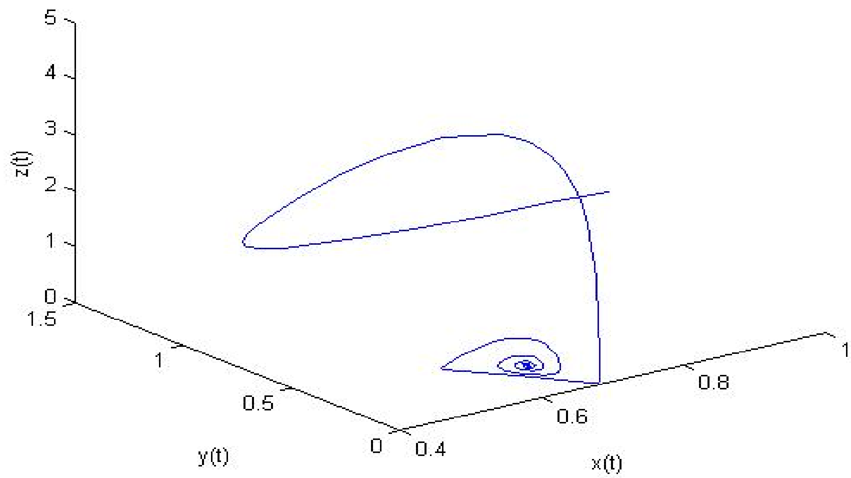

FIGURE 6. Phase diagram showing stability behavior of (2.1) around the equilibrium point $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$.

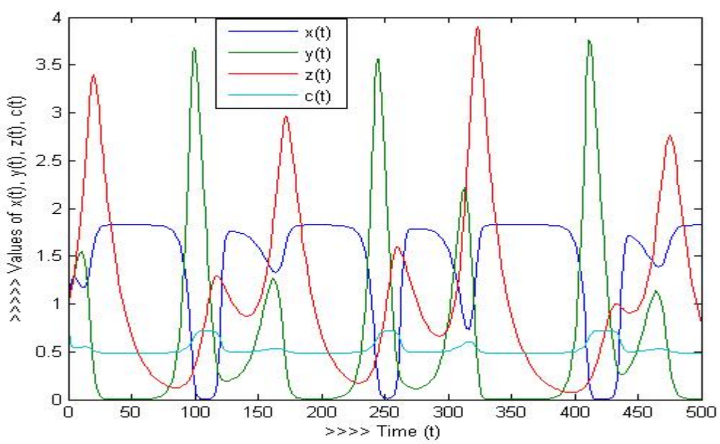

FIGURE 7. Unstable behavior of (2.1) around the equilibrium point $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$.

[5] Manaqib, M., Suma'inna & Zahra, A. (2022). Mathematical Model of Three Species Food Chain with Intraspecific Competition and Harvesting on Predator, *Barekeng: Jurnal Ilmu Matematika dan Terapan* **16(2)**, 551-562. https://doi.org/10.30598/barekengvol16iss2pp551–562 1

[6] Subbey, S., Frank, A. S., & Kobras, M. (2020). Crowding Effects in an Empirical Predator-Prey System, *bioRxiv*, 2020-08. https://doi.org/10.1101/2020.08.23.263384 1

[7] Zaowang X., Xiangdong X., & Yalong X., (2018). Stability and bifurcation in a Holling type II predator–prey model with Allee effect and time delay, Advances in Difference Equations volume 2018, Article number: 288, 1-21. 1

[8] Stollenwerk, N., Aguiar, M., & Kooi, B.W. (2022). Modelling Holling type II functional response in deterministic and stochastic food chain models with mass conservation, Ecological Complexity Volume 49, March, 100982 1

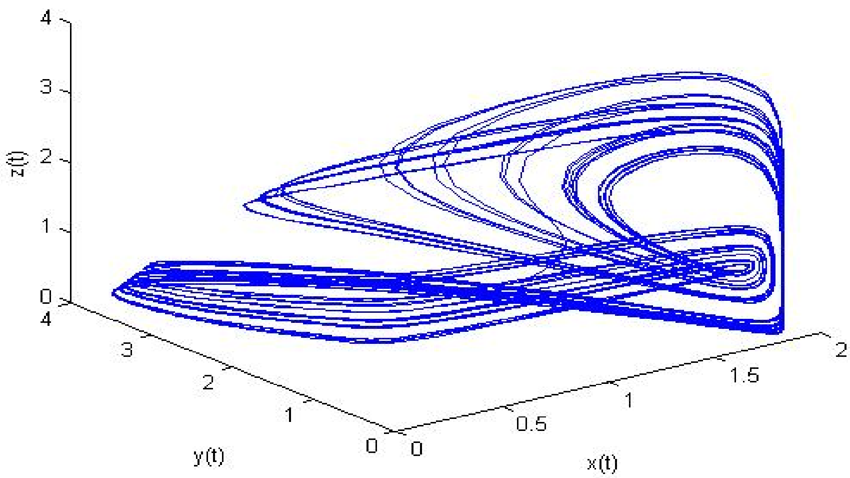

FIGURE 8. Phase diagram showing unstable behavior of (2.1) around the equilibrium point $\bar{E}_3(\bar{x}, \bar{y}, \bar{z}, \bar{c})$.

[9] Yusrianto, Toaha S & Kasbawati, (2019). Stability analysis of prey predator model with Holling II functional response and threshold harvesting for the predator, Journal of Physics: Conference Series 1341, 062025. 1

[10] Chattopadhyay, J. & Sarkar, R.R. (2003). Chaos to order: preliminary experiments with a population dynamics models of three trophic levels, Ecol. Mod. 163, 45–50. 1

[11] Peet, A.B., Deutsch, P.A., & Lo pez, E.P. (2005). Complex dynamics in a three-level trophic system with intraspecies interaction, J. of Theor. Bio. 232, 491–503. 1

[12] Kadhim, A. J., & Azhar, A. M. (2020). The impact of toxicant on the food chain ecological model, *AIP Conf. Proc.*, 2292(1).
https://doi.org/10.1063/5.0030690 1

[13] Misra, O. P. & Babu, A. R. (2017). Modelling the Effect of Toxicant on a Three Species Food-Chain System with Predator Harvesting, *Int. J. Appl. Comput.* **3**, 71-97.
DOI 10.1007/s40819-017-0342-4 1

[14] Misra, O.P. & Annavarapu, R. B. (2016). A model for the dynamical study of food-chain system considering interference of top predator in a polluted environment, *J. Math. Model.* **3(2)**, 189-218.
https://jmm.guilan.ac.ir/article_1471.html 1

[15] Waters, Edward, (2014). Modelling crowding effects in infectious disease transmission, *Thesis.* DOI: https://doi.org/10.26190/unsworks/17156 1

[16] Wang, C. C. & Liu, C. C. (2014). Chaotic Dynamic Analysis of Aquatic Phytoplankton System, Hindawi Publishing Corporation, Math. Prob. in Eng., Article ID 586262, 8 pages. 1

[17] Raveendra B., & Gayathri P., (2023). Mathematical Study of One Prey and Two Competing Predators Considering Beddington-DeAngelis Functional Response with Distributed Delay, Hilbert Journal of Mathematical Analysis, Volume 2 Number 1, Pages 001–019.
$https://hilbertjma.org/hilbertjma/article/view/14$ 1

[18] Raveendra B., Srajan G., Nisha R., Tanishka A., & Mamta S., (2024). The Effects of Crowding and Toxicant on Biological Food - Chain System: A Mathematical Approach, Hilbert Journal of Mathematical Analysis, Volume 2 Number 2, Pages 80–91.
$https://doi.org/10.62918/hjma.v2i2.24$ 1

[19] Raveendra B., & Imtiyaz A. W., (2024). Dynamics of Interaction of One Prey and Two Competing Predators with Population Heterogeneity, Indonesian Journal of Mathematics and Applications, 2, 78 - 88.
$https://doi.org/10.21776/ub.ijma.2024.002.02.2$ 1