Parallelogram law of an inner product space by substituting y = mx + c in the differential equation

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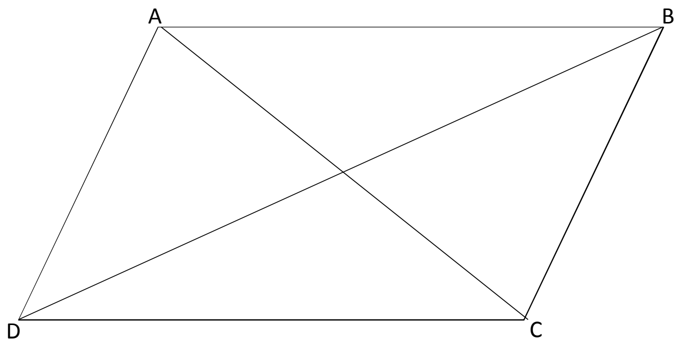
Abstract

In this paper, we present a parallelogram law of an inner product space by substituting y = mx+ c in the differential equations likely Hermite's differential equation, Legendre's differential equation and Laguerre's differential equation. This is the first report for the derivation of parallelogram law by substituting the equation of straight line y = mx + c in the above equations by introducing real linear space R and llxll = lxl , $x \in R$.

Keywords: Norm; Inner product space; Parallelogram law; Differential equation

1. Introduction

In 1935, Frechet gave a geometric characterization of an inner product space. In the same year, Jordan and von Neumann characterized the inner product spaces as a normed linear space satisfying the parallelogram law [1].



In the parallelogram ABCD, let AD = BC = α and AB = CD = β , $\angle$BAD = Φ

Now, using the law of cosines in ΔBAD, we get

$\alpha^2 + \beta^2 - 2\alpha\beta\cos(\alpha) = BD^2$ …………(i)

Since, in the parallelogram, the adjacent angles are supplementary

So, $\Delta ADC = 180^0 - \Phi$

Now, in ΔADC,

$\alpha^2 + \beta^2 - 2\alpha\beta\cos(180^0 - \Phi) = AC^2$

$\alpha^2 + \beta^2 + 2\alpha\beta\cos(\Phi) = AC^2$ ....(ii)

Adding (i) and (ii), we get

$\alpha^2 + \beta^2 - 2\alpha\beta\cos(180^0 - \Phi) + \alpha^2 + \beta^2 + 2\alpha\beta\cos(\Phi) = BD^2 + AC^2$

$\Rightarrow 2\alpha^2 + 2\beta^2 = BD^2 + AC^2$

$\Rightarrow BD^2 + AC^2 = 2\alpha^2 + 2\beta^2$

Hence, for any parallelogram, the sum of the squares of the lengths of its two diagonals is equal to the sum of the squares of the lengths of its four sides

i.e., $AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$

The parallelogram law has natural geometric interpretations, involving the areas of the squares constructed on the sides and on the diagonals of the parallelogram.

2. Definition

Let x be a vector space over the field F.Then a norm on x is a function$\|x\|:\rightarrow R$ such that

(a) $\|x\| \geq 0$ for all $x \in X$,

(b) $\|x\| = 0 \Rightarrow x = 0$

(c) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$, $\alpha \in F$

(d) $\|x + y\| \leq \|x\| + \|y\|$ for all $x , y \in X$

2.1. Theorem (Parallelogram Law)

If x and y are two vectors of a Hilbert space H then

$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Proof: $\|x + y\|^2 + \|x - y\|^2$

$= (x + y, x + y) + (x - y, x - y)$

$= (x, x) + (x, y) + (y, x) + (y, y) + (x, x) - (x, y) - (y, x) + (y, y)$

$= \|x\|^2 + \|y\|^2 + \|x\|^2 + \|y\|^2$

$= 2\|x\|^2 + 2\|y\|^2$

Definition.1 The differential equation of the form $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0$ where λ is a constant, is called Hermite's differential equation.

Definition.2 The differential equation of the form $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ is called Legendre's differential equation, where n is a constant.

Definition.3 The differential equation of the form $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0$ where λ is a constant,is called Laguerre's differential equation.

3. Discussions and Result :

3.1. Consider the Hermite's differential equation

$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0$, (λ is a constant) ……………...…(i)

Put $y = mx + c$ ………………………………………….(ii)

$\frac{dy}{dx} = m$ and $\frac{d^2y}{dx^2} = 0$

Using the above facts in (i), we get

$0 - 2xm + 2\lambda(mx + c) = 0$

$\Rightarrow -2xm + 2\lambda mx + 2\lambda c = 0$

$\Rightarrow -xm + \lambda mx + \lambda c = 0$

$\Rightarrow x(-m + \lambda m) = -\lambda c$

$\Rightarrow mx(\lambda - 1) = -\lambda c$

$\Rightarrow x = \frac{-\lambda c}{m(\lambda-1)}$, $m \neq 0, \lambda \neq 1$

Using this fact in (ii), we get

$y = \frac{-m\lambda c}{m(\lambda-1)} + c$

$\Rightarrow y = \frac{-\lambda c}{\lambda-1} + c$

$\Rightarrow y = \frac{-\lambda c + \lambda c - c}{\lambda-1}$

$\Rightarrow y = \frac{-c}{\lambda-1}$, $\lambda \neq 1$

We define real linear space R and the norm defined by

$\|x\| = |x|, x \in R$ ………….(α)

$\|x\|^2 = \left|\frac{-\lambda c}{m(\lambda-1)}\right|^2$

$= \lambda^2c^2 / m^2(\lambda-1)^2$ ……………………….(iii)

Also, $\|y\|^2 = \left|\frac{-c}{\lambda-1}\right|^2$

$= \frac{c^2}{(\lambda-1)^2}$ ………………………….(iv)

Now , $x + y = \frac{-\lambda c}{m(\lambda-1)} + \frac{-c}{\lambda-1}$

$= \frac{-\lambda c - mc}{m(\lambda-1)}$

Similarly , $x - y = \frac{-\lambda c + mc}{m(\lambda-1)}$

$\|x+y\|^2 = \left| \frac{-\lambda c - mc}{m(\lambda-1)} \right|^2$

$= \frac{(-\lambda c - mc)^2}{\{m(\lambda-1)\}^2}$

$= \frac{(\lambda c + mc)^2}{\{m(\lambda-1)\}^2}$

Similarly , $\|x-y\|^2 = \frac{(-\lambda c + mc)^2}{\{m(\lambda-1)\}^2}$

$\|x+y\|^2 + \|x-y\|^2 = \frac{(\lambda c + mc)^2}{\{m(\lambda-1)\}^2} + \frac{(-\lambda c + mc)^2}{\{m(\lambda-1)\}^2}$

$= \frac{(\lambda c + mc)^2 + (-\lambda c + mc)^2}{\{m(\lambda-1)\}^2}$

$= \frac{\lambda^2 c^2 + 2\lambda mc^2 + m^2 c^2 + \lambda^2 c^2 - 2\lambda mc^2 + m^2 c^2}{\{m(\lambda-1)\}^2}$

$= \frac{2\lambda^2 c^2 + 2m^2 c^2}{\{m(\lambda-1)\}^2}$

$= \frac{2\lambda^2 c^2}{\{m(\lambda-1)\}^2} + \frac{2m^2 c^2}{\{m(\lambda-1)\}^2}$

$= 2\frac{\lambda^2 c^2}{\{m(\lambda-1)\}^2} + 2\frac{m^2 c^2}{\{m(\lambda-1)\}^2}$

$= 2\frac{\lambda^2 c^2}{\{m(\lambda-1)\}^2} + 2\frac{m^2 c^2}{m^2(\lambda-1)^2}$

$= 2\frac{\lambda^2 c^2}{\{m(\lambda-1)\}^2} + 2\frac{c^2}{(\lambda-1)^2}$

$= 2\|x\|^2 + 2\|y\|^2$, (using (iii) and (iv))

Hence , $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

4.2. Consider the Legendre's differential equation

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ ……………………….(v)

Put y = mx + c in (v)

$\frac{dy}{dx} = m$ and $\frac{d^2y}{dx^2} = 0$

Using these facts in (v) , we get

-2mx + n(n+1)(mx+c) = 0

⇒ -2mx + n(n+1)mx + n(n+1)c = 0

⇒ m[n(n+1)x -2x] = -n(n+1)c

⇒ - [2m - n(n+1)]x = -n(n+1)c

$\Rightarrow x = n(n+1)c /\{ 2m - n(n+1)m\}$

$\Rightarrow x = n(n+1)c / m\{2 - n^2 - n\}$

$\Rightarrow x = - n(n+1)c/ m\{n^2 + n - 2 \}$

$\Rightarrow x = - n(n+1)c/ m\{n^2 + 2n - n - 2\}$

$\Rightarrow x = -n(n+1)c/ m(n-1)(n+2)$

Since , $y = mx + c$

$\Rightarrow y = -mn(n+1)c/ m(n-1)(n+2) + c$

$\Rightarrow y = \{- n(n+1)c + (n-1)(n+2)c\} / (n-1)(n+2)$

$\Rightarrow y = c \{ -n^2 - n + n^2 + 2n - n - 2\}/ (n-1)(n+2)$

$\Rightarrow y = -2c/ (n-1)(n+2)$

Now

$x + y = -n(n+1)c / m(n-1)(n+2) - 2c / (n-1)(n+2)$

$= c\{- n^2 - n - 2m\}/ m(n-1)(n+2)$

$= -c \{ 2m + n(n+1)\}/ m(n-1)(n+2)$

Also , $x - y = - n(n+1)c/ m(n-1)(n+2) + 2c/ (n-1)(n+2)$

$= c \{ 2m - n(n+1)\}/ m(n-1)(n+2)$

Since , $\|x\|^2 = |x|^2$,(using (α))

$= n^2 (n+1)^2 c^2 / m^2 (n-1)^2 (n+2)^2$

Aiso , $\|y\|^2 = |y|^2$

$= 4c^2 / (n-1)^2 (n+2)^2$

We have, $\|x + y\|^2 + \|x - y\|^2$

$= c^2 \{ 2m + n(n+1)\}^2/ m^2(n-1)^2(n+2)^2 + c^2 \{ 2m - n(n+1)\}^2/ m^2(n-1)^2(n+2)^2$

$$= \frac{c^2\{4m^2 + 4mn(n+1)+n^2(n+1)^2+4m^2 - 4mn(n+1)+n^2(n+1)^2\}}{m^2(n-1)^2(n+2)^2}$$

$$= \frac{8c^2}{(n-1)^2(n+2)^2} + \frac{2n^2(n+1)^2c2}{m^2(n-1)^2(n+2)^2}$$

$$= 2\frac{4c^2}{(n-1)^2(n+2)^2} + 2\frac{n^2(n+1)^2c2}{m^2(n-1)^2(n+2)^2}$$

$$= 2\frac{n^2(n+1)^2c2}{m^2(n-1)^2(n+2)^2} + 2\frac{4c^2}{(n-1)^2(n+2)^2}$$

$= 2 \|x\|^2 + 2 \|y\|^2$

Hence, $\|x + y\|^2 + \|x - y\|^2 = 2 \|x\|^2 + 2 \|y\|^2$

4.3. Consider the Laguerre's differential equation

$x \frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0$ ………………(i)

Put y = mx + c .

Then , $\frac{dy}{dx} = m$ and $\frac{d^2y}{dx^2} = 0$

Using the above facts in (i) , we get

$(1-x)m + \lambda(mx+c) = 0$

$\Rightarrow m - mx + m\lambda x + c\lambda = 0$

$\Rightarrow x(m\lambda - m) = -(m + c\lambda)$

$\Rightarrow x(m - m\lambda) = m + c\lambda$

$\Rightarrow x = \frac{m+c\lambda}{m-m\lambda}$

Since, y= mx+c

$\Rightarrow y = m\left(\frac{m+c\lambda}{m-m\lambda}\right) + c$

$\Rightarrow y = \frac{m^2 + mc\lambda + mc - mc\lambda}{(m-m\lambda)}$

$\Rightarrow y = \frac{m(m+c)}{m(1-\lambda)}$

$\Rightarrow y = \frac{m+c}{1-\lambda}$

Now , from (α), we get

$||x||^2 = |x|^2$

$= |\frac{m+c\lambda}{m-m\lambda}|^2$

$= \frac{(m+c\lambda)^2}{(m-m\lambda)^2}$

Also, $||y||^2 = \frac{(m+c)^2}{(1-\lambda)^2}$

We have, $\|x + y\|^2 = \left| \frac{m+c\lambda}{m-m\lambda} + \frac{m+c}{1-\lambda} \right|^2$

$= \left(\frac{m+c\lambda}{m-m\lambda} + \frac{m+c}{1-\lambda}\right)^2$

$= \left(\frac{m+c\lambda}{m-m\lambda}\right)^2 + \frac{2(m+c\lambda)(m+c)}{(m-m\lambda)(1-\lambda)} + \left(\frac{m+c}{1-\lambda}\right)^2$

Also, $\|x - y\|^2 = \left(\frac{m+c\lambda}{m-m\lambda} - \frac{m+c}{1-\lambda}\right)^2$

$= \left(\frac{m+c\lambda}{m-m\lambda}\right)^2 - \frac{2(m+c\lambda)(m+c)}{(m-m\lambda)(1-\lambda)} + \left(\frac{m+c}{1-\lambda}\right)^2$

Therefore , $\|x + y\|^2 + \|x - y\|^2 = 2\left(\frac{m+c}{m-m\lambda}\right)^2 + 2\left(\frac{m+c}{1-\lambda}\right)^2$

$= 2\|x\|^2 + 2\|y\|^2$

Hence, $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Conclusion : Hence, by substituting y = mx+ c in above differential equations we get parallelogram law i.e $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Reference:

[1] P. Jordan , J. von Neumann , on inner products in linear , metric spaces, Ann. Of Math.(2) 36 (3) (1935) 719-723.

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