Boundary Value Problems for Neutral Fractional Integro-Differential Equations with ψ-Caputo Fractional and Ulam Stability

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Abstract

This article examines the existence, uniqueness, and Ulam-Hyers stability(U-HS) of neutral fractional integro-differential equations(NFIDEs) in a Banach space that use ψ-Caputo fractional derivatives(ψ-C-FDs). The Banach Contraction Mapping Principle (BCMP) and Krasnoselskii's Fixed Point Theorem (KFPT) are used to determine the existence and uniqueness of solutions. A well-crafted example is presented to demonstrate the practicality of the theoretical ideas, improving and expanding upon earlier scholarly efforts.
Keywords: ψ-Caputo fractional derivatives, fractional integro differential equation, existence and uniqueness, fixed point theorems.
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1 Introduction

Fractional calculus extends the well-known ideas of differentiation and integration beyond integer-order derivatives and integrals to arbitrary (real or complex) orders. In recent years, this topic has drawn a lot of attention, especially for simulating different processes in mathematical engineering, biology, physics, and other fields. Fractional differential operators offer a more accurate and nuanced depiction of numerous real-world processes than conventional differential operators, which are local in nature.The works of Abbas et al. [1], Hilfer [14], Kilbas et al. [17], Miller and Ross [21], Pudlubny [25], Tarasov [31], and their references are recommended reading for those interested in the theory of fractional differential equations. These papers and monographs offer a more thorough comprehension and advancements in the topic of fractional calculus. The theory combines pure and practical analysis in a lovely way. In the study of nonlinear processes, the idea of fixed points has shown to be a very useful and significant tool over time.

Electromagnetism, population dynamics, fluid mechanics, heat conduction with memory, control theory, and other physical phenomena involving memory effects, genetic processes, or systems where the current state depends on the cumulative history of previous states are all frequently modeled using integro-differential equations, which combine differential and integral equations; see that [3, 9, 18, 24].

The existence and uniqueness of solutions for the following nonlinear ψ-integral differential equation with nonlocal boundary condition and variable coefficients are examined by the authors in [4].

$$
\begin{aligned}
&{}^{C}\mathfrak{D}^{\eta,\psi}\mathcal{Y}(\kappa) + \mathcal{A}(\kappa)\mathrm{I}^{\beta;\psi}\mathcal{Y}(\kappa) = \mathcal{G}(\kappa,\mathcal{Y}(\kappa)), \quad \kappa \in [\mathcal{A},\mathfrak{b}],\\
&\mathcal{Y}(\mathcal{A}) = -\mathcal{G}(\mathcal{Y}),\ \mathcal{Y}'(\mathcal{A}) = \ldots = \mathcal{Y}^{i}(\mathcal{A}) = 0,\\
&\int_{\mathcal{A}}^{\mathfrak{b}} \psi'(\kappa)\mathcal{Y}(\kappa)\mathrm{d}\kappa = \varsigma,
\end{aligned}
$$

where $\mathcal{A}(\kappa)$-is a variable coefficients, ς-is constants and ${}^{C}\mathfrak{D}^{\eta,\psi}$ is ψ-Riemann Lioville fractional integral operators and $\mathrm{I}^{\beta;\psi}$-is the ψ-Caputo fractional.

The authors of [6] study a fractional functional differential equation of the type:

$$
\begin{aligned}
&{}^{C}\mathfrak{D}^{\eta;\psi}\mathcal{Y}(\kappa) = \mathcal{G}(\kappa,\mathcal{Y}_{\kappa}), \quad \kappa \in [\mathfrak{a},\mathfrak{b}],\\
&\mathcal{Y}(\kappa) = \phi(\kappa), \quad \kappa \in [\mathfrak{a}-\epsilon,\mathfrak{a}],
\end{aligned}
$$

where ${}^{C}\mathcal{D}^{\eta;\psi}$is the ψ-Caputo fractional derivative.

Recent research has focused a lot of attention on the existence and uniqueness of solutions to fractional differential equations. Further information can be found in previous studies [7, 11, 20, 29] and their references. However, approximate solutions are also taken into consideration because it is frequently difficult to obtain exact solutions, especially in nonlinear analysis and optimization. It is important to stress that only stable approximations are considered appropriate. As a result, different stability analysis methods are used. A popular strategy is the idea of Hyers Ulam-type stability, which is simple and has been covered in great detail in the literature. Ulam first suggested this kind of stability, which Hyers expanded upon the following year. This idea was first used to solve ordinary differential equations, and then it was expanded to include fractional differential equations (FDEs).

The coupled hybrid system for the thermostat model involving ψ-C-FDs is examined by the authors in [15]:

$$
\begin{aligned}
&{}^{C}\mathfrak{D}^{\eta_1;\psi}\left(\frac{\mathcal{Y}_1(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)+\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))=0,\ \ \kappa\in[\mathfrak{a},\mathfrak{b}],\\
&{}^{C}\mathfrak{D}^{\eta_2;\psi}\left(\frac{\mathcal{Y}_2(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)+\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))=0,\ \ \kappa\in[\mathfrak{a},\mathfrak{b}],
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathfrak{D}\left(\frac{\mathcal{Y}_1(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)=\mathfrak{D}\left(\frac{\mathcal{Y}_2(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)=0,\\
&\alpha_1^{C}\mathfrak{D}^{\eta_1-1;\psi}\left(\frac{\mathcal{Y}_1(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)+\left(\frac{\mathcal{Y}_1(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)=0,\\
&\alpha_2^{C}\mathfrak{D}^{\eta_1-1;\psi}\left(\frac{\mathcal{Y}_2(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)+\left(\frac{\mathcal{Y}_2(\kappa)}{\mathrm{g}(\kappa,\mathcal{Y}_1(\kappa),\mathcal{Y}_2(\kappa))}\right)=0,
\end{aligned}
$$

where $1 > \eta_i \leq 2$, α_i-is a +ve real number, and ${}^{C}\mathfrak{D}^{\eta_1;\psi}, {}^{C}\mathfrak{D}^{\eta_2;\psi}$-denotes the ψ-C-FDs of order η_i and η_{i-1}.

Due to its many applications in science, boundary value problems(BVPs) involving fractional differential equations have recently drawn a lot of attention from academics and are becoming a significant area of study. These applications include, among other things, control theory, mechanics, biology, and wave propagation [33].

When modeling complex systems seen in biology, physics, and engineering, neutral functional differential equations (NFDEs) are essential, as demonstrated in [2, 8, 9]. Initial value problems (IVPs) utilizing fractional NFDEs with finite delays are particularly notable because of their inherent capacity to capture memory effects and genetic traits. As reported in [18, 19, 26], many academics have created a range of analytical methods to explore qualitative aspects of such equations employing classical FODs.

The authors of [16] look at the following ψ-the Caputo fractional integro-differential equation is as follows:

$$\begin{aligned}&{}^{C}\mathfrak{D}^{\eta;\psi}\mathcal{Y}(\kappa)=\mathcal{G}(\kappa,\mathcal{Y}(\kappa),\mathfrak{B}\mathcal{Y}(\kappa)),\ \kappa\in[0,\mathcal{T}],\\ &\mathcal{Y}(\mathcal{T})=\nu\mathcal{Y}(\gamma),\end{aligned}$$

where ${}^{C}\mathcal{D}^{\eta;\psi}$is the ψ-Caputo fractional derivative.

Inspired by past studies, the project intends to investigate NFIDs using ψ-C-FDs with BVPs:

$$\begin{aligned}&{}^{C}\mathfrak{D}^{\eta;\psi}[\mathcal{Y}(\kappa)-\Pi(\kappa,\mathcal{Y}(\kappa))]=\mathcal{G}(\kappa,\mathcal{Y}(\kappa),\mathfrak{B}\mathcal{Y}(\kappa)),\ 2<\eta\leq 3,\ \kappa\in \mathrm{J}:[0,1], \qquad (1)\\ &\mathcal{Y}(0)=\mathcal{Y}'(0)=0,\ {}^{C}\mathfrak{D}^{\eta-1;\psi}\mathcal{Y}(1)=1,\end{aligned}$$

where ${}^{C}\mathcal{D}^{\eta;\psi}$is the ψ-Caputo fractional derivative and $\Pi:[0,1]\times[0,\infty)\to\mathbb{R}$, $\mathcal{G}:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous. Moreover, $\mathfrak{B}\mathcal{Y}(\kappa)=\int_0^{\kappa}\mathcal{K}(\kappa,\lambda,\mathcal{Y}(\lambda))\mathrm{d}\lambda$ and $\mathcal{K}\in\mathcal{C}(\mathfrak{D},\mathbb{R}^+)$ $\quad\mathfrak{D}=\{(\kappa,\lambda)\in\mathbb{R}^2:0\leq\lambda\leq\kappa\leq 1\}$.

Based on the above-mentioned reasons, our study tries to bridge the existing research gap. Below we briefly describe the key contributions of this study.

I) There is lack of literature due to the limited amount of study conducted on the study of U-HS. In this paper, the main purpose will be the study of U-HS of NFIDEs including the ψ-C-FDs in a Banach space.

II) The U-HS will be obtained, and the existence and uniqueness of solutions of NFIDEs with the ψ-C-FDs will be proved through the help of BCMP and KFPT.

III) In order to emphasize the use of the results numerically found for solving real life problems, an example will also be provided.

The remainder of the paper will be organized in the following manner. Section 2 provides the introduction of lemmas and definitions necessary for proving the key results. Sections 3 investigate the E-UR of solutions for the problem formulations specified in Eq (1) . Section 4 discusses the application aspects of the obtained results.

2 Preliminaries

Let $\mathcal{C}(\mathrm{J}, \mathbb{R})$ denote the Banach space of all continuous functions form $\mathcal{T}$ into $\mathbb{R}$ with the norm $\|\mathcal{Y}\|_\infty = \sup\{|\mathcal{Y}(\kappa)| : \kappa \in \mathrm{J}\}$.

Definition 1 *The Caputo FD of order* $\eta > 0$, $\mathrm{n} - 1 < \eta < \mathrm{n}$, $(\mathrm{n} \in \mathbb{N})$*, is defined as*

$$ {}^{C}\mathfrak{D}^{\eta}_{0^+}\mathcal{G}(\kappa) = \mathrm{I}^{\mathrm{n}-\eta}\mathfrak{D}^{\mathrm{n}}\mathcal{G}(\kappa) = \frac{1}{\Gamma(\mathrm{n}-\eta)}\int_1^\kappa (\kappa-\lambda)^{\mathrm{n}-\eta-1}\mathcal{G}^{(\mathrm{n})}(\lambda)\mathrm{d}\lambda. $$

Definition 2 *The* ψ*-C-FDs of order* η *of a function* $\mathcal{G}$ *is defined as*

$$ \begin{aligned} {}^{C}\mathfrak{D}^{\eta;\psi}_{0^+}\mathcal{G}(\kappa) &= \mathrm{I}^{\mathrm{n}-\eta;\psi}\mathcal{G}^{[\mathrm{n}]}_{\theta}(\kappa), \\ &= \frac{1}{\Gamma(\mathrm{n}-\eta)}\int_{\mathfrak{a}}^{\kappa} \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\mathrm{n}-\eta-1}\mathcal{G}^{[\mathrm{n}]}_{\psi}\mathrm{d}\lambda, \end{aligned} $$

where $\mathrm{n} - [\eta] + 1$*, for* $\eta \notin \mathbb{N}$ *and* $\mathcal{G}^{[\mathrm{n}]}_{\theta}(\kappa) = \left(\frac{1}{\psi'(\kappa)}\frac{\mathrm{d}}{\mathrm{d}\kappa}\right)^{\mathrm{n}}\mathcal{G}(\kappa)$ *on* $[\mathfrak{a}, \mathfrak{b}]$*. Let* $\mathrm{n} \in \mathbb{N}, \mathcal{G}, \psi \in \mathcal{C}^n([\mathfrak{a}, \mathfrak{b}])$ *be two functions such that* ψ *is increasing with* $\psi'(\kappa) \neq 0$*, for all* $\kappa \in [\mathfrak{a}, \mathfrak{b}]$*.*
From the definition, it is clear that where $\eta = \mathrm{n} \in \mathbb{N}$*, we have*

$$ {}^{C}\mathfrak{D}^{\eta;\psi}_{0^+}\mathcal{G}(\kappa) = \mathcal{G}^{[\mathrm{n}]}_{\theta}(\kappa). $$

We note that if $\mathcal{G} \in ([\mathfrak{a}, \mathfrak{b}])$ *and* $\eta > 0$*. Then we have*

$$ {}^{C}\mathfrak{D}^{\eta;\psi}_{0^+}\mathcal{G}(\kappa) = C\mathfrak{D}^{\eta;\psi}_{0^+}\left(\mathcal{G}(\kappa) - \sum_{\mathrm{k}=0}^{\mathrm{n}-1}\frac{\mathcal{G}^{[\mathrm{k}]}_{\psi}(\mathfrak{a}^+)}{\mathrm{k}!}(\psi(\kappa)-\psi(\mathfrak{a})^{\mathrm{k}})\right). $$

Theorem 3 *Let* $\mathcal{G} \in ([\mathfrak{a}, \mathfrak{b}])$*. The* ψ*-C-FDs of order* η *of* $\mathcal{G}$ *is determined as*

$$ \mathrm{I}^{\eta;\psi}\ {}^{C}\mathfrak{D}^{\eta;\psi}_{0^+}\mathcal{G}(\kappa) = \mathcal{G}(\kappa) - \sum_{\mathrm{k}=0}^{\mathrm{n}-1}\frac{\mathcal{G}^{[\mathrm{k}]}_{\psi}(\mathfrak{a}^+)}{\mathrm{k}!}(\psi(\kappa)-\psi(\mathfrak{a})^{\mathrm{k}}). $$

In particular, given $\eta \in (0, 1)$ *we have*

$$ \mathrm{I}^{\eta;\psi}\ {}^{C}\mathfrak{D}^{\eta;\psi}_{0^+}\mathcal{G}(\kappa) = \mathcal{G}(\kappa) - \mathcal{G}(\mathfrak{a}). $$

Theorem 4 *Given a function* $\mathcal{G} \in ([\mathfrak{a}, \mathfrak{b}])$ *and* $\eta > 0$*, we have*

$$ {}^{C}\mathfrak{D}^{\eta-1;\psi}_{\mathfrak{a}^+}\ \mathrm{I}^{\eta;\psi}\ \mathcal{G}(\kappa) = \int_{\mathfrak{a}}^{\kappa}\mathcal{G}(\kappa)\psi'(\kappa)\mathrm{d}\kappa. $$

Lemma 5 *Given* $\mathrm{n} \leq \mathrm{k} \in \mathbb{N}$*, we have*

$$ {}^{C}\mathfrak{D}^{\eta;\psi}_{\mathfrak{a}^+}(\psi(\kappa)-\psi(\mathfrak{a}))^{\mathrm{k}} = \frac{\mathrm{k}!}{\Gamma(\mathrm{k}+1-\eta)}(\psi(\kappa)-\psi(\mathfrak{a}))^{\mathrm{k}-\eta} $$

and

$$ {}^{C}\mathfrak{D}^{\eta;\psi}_{\mathfrak{b}^-}(\psi(\mathfrak{b})-\psi(\kappa))^{\mathrm{k}} = \frac{\mathrm{k}!}{\Gamma(\mathrm{k}+1-\eta)}(\psi(\mathfrak{b})-\psi(\kappa))^{\mathrm{k}-\eta}. $$

Lemma 6 *A function* $\mathcal{X} \in \mathrm{C}([1,\mathcal{T}],\mathbb{R})$ *of the problem (1) is given by,*

$$ \begin{aligned} &{}^{C}\mathfrak{D}^{\eta;\psi}[\mathcal{Y}(\kappa)-\Pi(\kappa,\mathcal{Y}(\kappa))] = \mathcal{G}(\kappa),\ 2<\eta\leq 3,\ \kappa\in \mathrm{J}:[0,1], \\ &\mathcal{Y}(0)=\mathcal{Y}'(0)=0,\ {}^{C}\mathfrak{D}^{\eta-1;\psi}\mathcal{Y}(1)=1, \end{aligned} \tag{2} $$

if and only if

$$ \begin{aligned} \mathcal{Y}(\theta) = {} & \Pi(\theta,\mathcal{X}(\theta))\mathcal{Y}(\theta) + \frac{-\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^{\eta} \\ & \times\left(\int_0^1 \mathcal{G}(\lambda)\psi'(\lambda)\mathrm{d}\lambda\right) \\ & + \frac{1}{\Gamma(\eta)}\int_0^{\kappa}\psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda)\mathrm{d}\lambda. \end{aligned} \tag{3} $$

Proof. Applying the ψ-Riemann-Liouville fractional integral of order η to the first equation of(),

$$ \begin{aligned} \mathcal{Y}(\kappa) = {} & \mathrm{C}_0 + \mathrm{C}_1(\psi(\kappa)-\psi(0)) + \mathrm{C}_2(\psi(\kappa)-\psi(0))^{\eta} + \Pi(\kappa,\mathcal{Y}(\kappa)) \\ & + \frac{1}{\Gamma(\eta)}\int_0^{\kappa}\psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda)\mathrm{d}\lambda, \end{aligned} $$

Applying the first boundary condition $\mathcal{Y}(0)=0$ and $\mathcal{Y}'(0)=0$, we deduce that $\mathrm{C}_0=\mathrm{C}_1=0$. Then

$$ \begin{aligned} \mathcal{Y}(\kappa) = {} & \mathrm{C}_2(\psi(\kappa)-\psi(0))^{\eta} + \Pi(\kappa,\mathcal{Y}(\kappa)) \\ & + \frac{1}{\Gamma(\eta)}\int_0^{\kappa}\psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda)\mathrm{d}\lambda, \end{aligned} $$

with the condition ${}^{C}\mathfrak{D}^{\eta-1;\psi}_{\mathfrak{a}^+}\mathcal{G}(\kappa)=1$ and theorem (2.2), we have

$$ {}^{C}\mathfrak{D}^{\eta-1;\psi}_{\mathfrak{a}^+}\mathcal{G}(\kappa) = \frac{2\mathrm{C}_2}{\Gamma(4-\eta)}(\psi(\kappa)-\psi(0))^{3-\eta}, $$

and

$$\mathrm{C}_2 = -\left(\int_0^1 \mathcal{G}(\lambda)\psi'(\lambda)\mathrm{d}\lambda\right)\frac{\Gamma(4-\eta)}{2}(\psi(1)-\psi(0))^{\eta-3}.$$

Thus, we get the integral equation (3) and the converse follows by direct computation which completes the proof. □

Theorem 7 (Banach Fixed Point Theorem) *Let (X, d) be a complete metric space and $T : X \to X$ be a function. T is a contraction if there exists a constant q such that for all $(x, y \in X) : (d(Tx, Ty) \leq d(x, y))$.*

Theorem 8 (Krasnoselskii's Fixed Point Theorem) *Let $\mathcal{K}$ be a closed convex, bounded and non empty subset of a Banach space $\mathcal{X}$. Let P_1, P_2, be two operators such that:*
(i)$P_1\mathfrak{z} + P_2 y \in \mathcal{K}$ for any $\mathfrak{z}, y \in \mathcal{K}$.
(ii) P_1 is completely continuous operator.
(iii)P_1 is contraction operator.
Then there exists one fixed point $\mathfrak{z} \in \mathcal{K}$ such that $\mathfrak{z} = P_1\mathfrak{z} + P_2\mathfrak{z}$ atleast.

3 Main Results

The analytical framework is based on the following hypotheses to validate the principal findings.
(H_1): F is a continuous function with some constants $K_1, K_2, N > 0$:
(i)$|\mathcal{F}(\theta, t_1, \mathrm{u}) - F(\zeta, t_2, \bar{\mathrm{u}})| \leq \mathrm{K}_1 |t_1 - t_2| + \mathrm{K}_2 |\mathrm{u} - \bar{\mathrm{u}}|, \ \forall t_1, t_2, \mathrm{u}, \bar{\mathrm{u}} \in \mathbb{R}$.
(ii)$|\mathcal{X}(\theta, \kappa, t_1) - \mathcal{X}(\theta, \kappa, t_2)| \leq \mathrm{N} |t_1 - t_2|$.
(H_2): $|\Pi(\theta, t_1) - \Pi(\theta, t_2)| \,\&\leq \mathrm{L_G} |t_1 - t_2|$, for $t_1, t_2 \in \mathbb{R}$.
(H_3): Let $\mathrm{K_F} > 0$ denote a constant: $|\mathcal{F}(\theta, t, \mathrm{v})| \leq \mathrm{K_F}$, $\forall\, \theta \in \mathcal{J}$ and $t, \mathrm{v} \in \mathbb{R}$.
(H_4): Let $\mathrm{L_F} > 0$ denote a constant: $|\Pi(\theta, t)| \,\&\leq \mathrm{L_F}\ t \in \mathbb{R}$.

Theorem 9 *Under the condition $(H_1 - H_4)$, the existence of at least one solution to the problem (1) on the interval $\mathcal{J}$ is guaranteed.*

Proof. Consider, $\mathrm{B}_{\mathcal{P}} = \{\mathcal{X} \in \mathfrak{PC}(\mathcal{J}, \mathbb{R}) : |\mathcal{X}| \leq \mathcal{P}\}$.
Define the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ on the ball $\mathrm{B_P}$ by

$$(\mathcal{P}_1\mathcal{Y})(\kappa) = \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda, \quad (4)$$

and

$$
\begin{aligned}
(\mathcal{P}_2\mathcal{Y})(\kappa) =&\Pi(\kappa,\mathcal{Y}(\kappa)) - \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2\\
&\times\left(\int_0^1 \mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\right)
\end{aligned}
\tag{5}
$$

If $\mathcal{Y},\mathcal{X}\in\mathrm{B_P}$, then $\mathcal{P}_1\mathcal{Y}+\mathcal{P}_2\mathcal{X}\in\mathrm{B_P}$. As we now show:

$$
\begin{aligned}
|\mathcal{P}_1\mathcal{Y}+\mathcal{P}_2\mathcal{X}| \leq&\Bigg\|\Pi(\kappa,\mathcal{Y}(\kappa)) - \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2\\
&\times\left(\int_0^1 \mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\right)\\
&+\frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\Bigg\|\\
&\leq|\Pi(\kappa,\mathcal{Y}(\kappa))| + \Bigg|\frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2\\
&\times\left(\int_0^1 \mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\right)\Bigg|\\
&+\left|\frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\right|,\\
&\leq|\Pi(\kappa,\mathcal{Y}(\kappa))| + \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2\\
&\times\left(\int_0^1 |\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))|\,\psi'(\lambda)\mathrm{d}\lambda\right)\\
&+\frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\,|\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))|\,\mathrm{d}\lambda,\\
&\leq\left[\mathrm{L_F}+(\mathrm{K_F})(\psi(1)-\psi(0))^\eta\left[\frac{\Gamma(4-\eta)}{2}+\frac{1}{\Gamma(\eta+1)}\right]\right],\\
&\leq\mathrm{P}.
\end{aligned}
$$

Thus $\mathcal{P}_1\mathcal{Y}+\mathcal{P}_2\mathcal{X}\in\mathrm{B_P},\ \ \mathcal{Y},\mathcal{X}\in\mathrm{B_P}$.
The continuity of $\mathcal{Y}$ and the operator $(\mathcal{P}_1\mathcal{Y})(\kappa)$ implies that $(\mathcal{P}_2)$ is a contrac-

tion. Moreover, observe that

$$\begin{aligned}|(\mathcal{P}_1\mathcal{Y})(\kappa)| &\leq \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\\ &\leq \frac{(\psi(1)-\psi(0))^\eta}{\Gamma(\eta+1)}\mathrm{K_F}.\end{aligned}$$

It follows that $(\mathcal{P}_1)$ remains uniformly bounded throughout the domain $\mathrm{B_P}$. Let $\kappa_1, \kappa_2 \in \mathrm{J}$ with $\kappa_2 \leq \kappa_1$, and $\mathcal{Y} \in \mathrm{B_P}$. Since $\mathcal{G}$ is bounded on compact sets,

$$(\mathcal{P}_1\mathcal{Y})(\kappa)$$

is equicontinuous on J. Since

$$\sup_{(\kappa,\mathcal{Y},\mathcal{X})\in \mathrm{J}\times\mathrm{B_P}} |\mathcal{G}(\kappa,\mathcal{Y}(\kappa),\mathrm{B}\mathcal{Y}(\kappa))| := \mathcal{C}_0 < \infty,$$

we will get

$$\begin{aligned}&|(\mathcal{P}_1\mathcal{Y})(\kappa_2)-(\mathcal{P}_1\mathcal{Y})(\kappa_1)|\\ &= \Big|\frac{1}{\Gamma(\eta)}\int_1^{\kappa_2} \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\\ &-\frac{1}{\Gamma(\eta)}\int_1^{\kappa_1} \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\Big|\\ &\leq \frac{1}{\Gamma(\eta)}\int_1^{\kappa_2} \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\left|\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\right|\\ &+\frac{1}{\Gamma(\eta)}\int_1^{\kappa_1} \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\left|\mathcal{G}(\lambda,\mathcal{Y}(\lambda),\mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\right|\\ &\leq \frac{1}{\Gamma(\eta)}\int_1^{\kappa_2} \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathrm{C}_0\mathrm{d}\lambda\\ &+\frac{1}{\Gamma(\nu)}\int_1^{\kappa_1} \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathrm{C}_0\mathrm{d}\lambda\\ &\leq \frac{\mathrm{C}_0}{\Gamma(\eta)}[2(\psi(\kappa_2)-\psi(0)^\eta+(\psi(\kappa_1)-\psi(0)^\eta] \quad \to 0 \quad \text{as} \quad \kappa_2\to\kappa_1.\end{aligned}$$

By the Ascoli-Arzela theorem, the relative compactness of $\mathcal{P}_1(\mathrm{B_P})$, established through its equicontinuity and boundedness, ensures that $\mathcal{P}_1$ is compact. Hence, the problem (1) has at least one fixed point on the interval J. □

Theorem 10 *Suppose that the assumptions* $(H_1)-(H_2)$ *and the following inequality hold:*

$$\Theta = \left[\mathrm{L_G}+(\mathrm{K}_1+\mathrm{K}_2\mathrm{N})(\psi(1)-\psi(0))^\eta\left[\frac{\Gamma(4-\eta)}{2}+\frac{1}{\Gamma(\eta+1)}\right]\right] < 1. \qquad (6)$$

Under these conditions, the problem (1) *possesses a unique solution over the interval* $\mathcal{J}$.

Proof. Consider the operator P defined by

$$\begin{aligned}(\mathcal{P}\mathcal{Y})(\kappa) = \Pi(\kappa, \mathcal{Y}(\kappa)) - &\frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\ &\times \left(\int_0^1 \mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\right) \\ &+ \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda.\end{aligned}$$

At this stage, our objective is to demonstrate that the operator $\mathcal{P}$ satisfies the contraction property.
Let $\mathcal{Y}, \mathcal{X} \in \mathbb{R}$ and $\kappa \in \mathrm{J}$ be given. Then we proceed as follows:

$$\begin{aligned}&|(\mathcal{P}\mathcal{Y})(\kappa) - (\mathcal{P}\mathcal{X})(\kappa)| \\ &\leq \Bigg|\Pi(\kappa, \mathcal{Y}(\kappa)) - \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\ &\times \left(\int_0^1 \mathcal{G}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\right) \\ &+ \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{G}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\Bigg|, \\ &\leq |\Pi(\kappa, \mathcal{Y}(\kappa)) - \Pi(\kappa, \mathcal{X}(\kappa))| + \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\ &\times \left(\int_0^1 |\mathcal{G}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda)) - \mathcal{G}(\lambda, \mathcal{X}(\lambda), \mathrm{B}\mathcal{X}(\lambda))|\, \psi'(\lambda)\mathrm{d}\lambda\right) \\ &+ \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1} \\ &|\mathcal{G}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda)) - \mathcal{G}(\lambda, \mathcal{X}(\lambda), \mathrm{B}\mathcal{X}(\lambda))|\, \mathrm{d}\lambda, \\ &\leq \mathrm{L_G}\, |\mathcal{Y}(\kappa) - \mathcal{X}(\kappa)| + \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\ &\times \left(\int_0^1 (\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})\, |\mathcal{Y}(\lambda) - \mathcal{X}(\lambda)|\, \psi'(\lambda)\mathrm{d}\lambda\right) \\ &+ \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}(\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})\, |\mathcal{Y}(\lambda) - \mathcal{X}(\lambda)|\, \mathrm{d}\lambda,\end{aligned}$$

$$\leq \left[\mathrm{L_G} + (\mathrm{K_1} + \mathrm{K_2 N}) \left[\frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta}}{2} + \frac{(\psi(1)-\psi(0))^{\eta}}{\Gamma(\eta+1)} \right] \right] \|\mathcal{Y} - \mathcal{X}\|,$$
$$\leq \left[\mathrm{L_G} + (\mathrm{K_1} + \mathrm{K_2 N})(\psi(1)-\psi(0))^{\eta} \left[\frac{\Gamma(4-\eta)}{2} + \frac{1}{\Gamma(\eta+1)} \right] \right] \|\mathcal{Y} - \mathcal{X}\|.$$

Since the coefficient of the last expression is less than one, the operator $\mathcal{P}$ is a contraction. Therefore, the problem (1) possesses a unique solution on the interval $[0, 1]$. □

4 Stability in the sense of Ulam

A system's stability is crucial for ensuring dependability in applications like transportation systems, which need to function smoothly and safely, both theoretically and practically.Mathematical stability is crucial to the analysis of DEs controlling such systems, which has been extensively studied by scholars and yielded valuable results (see [9, 10, 12, 13, 22, 27]). The U-HS stability method is widely used; see, for example, [28, 30]. See, for example, [34–37] and stability studies for various integro differential equations with delay [30â€“32]. U-HS of FDEs has a long history and good results. In general, it provides an answer to the question of how close an equation's estimated solution is to its precise solution.

Definition 11 *Problem (1) is called U-HS if some constant* $\Upsilon_{\mathcal{U}} > 0$ *exists such that for any* $\varsigma > 0$ *and any function* $\mathcal{Y}(\kappa)$ *satisfying*

$$\left| {}^{C}\mathfrak{D}^{\eta;\psi}[\mathcal{Y}(\kappa) - \Pi(\kappa, \mathcal{Y}(\kappa))] - \mathcal{G}(\kappa, \mathcal{Y}(\kappa), \mathfrak{B}\mathcal{Y}(\kappa)) \right| \leq \varsigma, \tag{7}$$

and there exists a solution $\mathcal{X}(\kappa)$ *such that*

$$|\mathcal{X}(\kappa) - \mathcal{Y}(\kappa)| \leq \Upsilon_{\mathcal{U}}\varsigma \tag{8}$$

Remark 12 *Assuming that* $\mathcal{Y} \in \mathrm{C}([0, 1], \mathbb{R})$ *and* $\mathrm{g}_i, (i = 1, 2, ..., k)$ *are sequences,* $\mathcal{Y}$ *satisfies inequality (8) whenever the corresponding conditions are should be:*

$(i)|\mathrm{g}(\kappa) \leq \varsigma, |\mathrm{g_k}| \leq \varsigma$;
$(ii){}^{C}\mathfrak{D}^{\eta;\psi}[\mathcal{Y}(\kappa) - \Pi(\kappa, \mathcal{Y}(\kappa))] = \mathcal{G}(\kappa, \mathcal{Y}(\kappa), \mathfrak{B}\mathcal{Y}(\kappa)) + \mathrm{g}(\kappa), \kappa \in [0, 1]$.

Lemma 13 *We consider the NFIDEs ψ-C-FDs with BVPs:*

$${}^{C}\mathfrak{D}^{\eta;\psi}[\mathcal{Y}(\kappa) - \Pi(\kappa, \mathcal{Y}(\kappa))] = \mathcal{G}(\kappa, \mathcal{Y}(\kappa), \mathfrak{B}\mathcal{Y}(\kappa)),\ 2 < \eta \leq 3,\ \kappa \in \mathrm{J} : [0, 1], \tag{9}$$

$$\mathcal{Y}(0) = \mathcal{Y}'(0) = 0,\ {}^{C}\mathfrak{D}^{\eta-1;\psi}\mathcal{Y}(1) = 1,$$

Then

$$\begin{aligned}\mathcal{Y}(\kappa) = \Pi(\kappa, \mathcal{Y}(\kappa)) &- \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\ &\times \left(\int_0^1 \mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\right) \\ &+ \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda. \end{aligned} \tag{10}$$

Moreover, from Lemma 6, we get

$$\begin{aligned}\Big|\mathcal{X}(\kappa) - \Pi(\kappa, \mathcal{Y}(\kappa)) &- \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\ &\times \left(\int_0^1 \mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\right) \\ &+ \frac{1}{\Gamma(\eta)}\int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\Big|, \\ &\leq \frac{(\psi(1)-\psi(0))^\eta}{\Gamma(\eta+1)}\varsigma. \end{aligned}$$

Theorem 14 *Le the conditons* $(H_1 - H_2)$ *and*

$$\begin{aligned} &{}^C\mathfrak{D}^{\eta;\psi}[\mathcal{Y}(\kappa) - \Pi(\kappa, \mathcal{Y}(\kappa))] = \mathcal{G}(\kappa, \mathcal{Y}(\kappa), \mathfrak{B}\mathcal{Y}(\kappa)),\ 2 < \eta \leq 3,\ \kappa \in \mathrm{J} : [0,1], \\ &\mathcal{Y}(0) = \mathcal{Y}'(0) = 0,\ {}^C\mathfrak{D}^{\eta-1;\psi}\mathcal{Y}(1) = 1, \end{aligned} \tag{11}$$

hold. Then the problem (1) *admits is U-H stability, if*

$$\vartheta : \left[\mathrm{L_G} + (\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})(\psi(1)-\psi(0))^\eta \left[\frac{\Gamma(4-\eta)}{2} + \frac{1}{\Gamma(\eta+1)}\right]\right] \leq 1. \tag{12}$$

Proof. Let $\mathcal{X} \in \mathrm{C}^1(\mathrm{J}, \mathbb{R})$ denote a solution of (10). Then,

$$
\begin{aligned}
&|\mathcal{X}(\kappa) - \mathcal{Y}(\kappa)| \\
&\leq \Big|\mathcal{X}(\kappa) - \Big(\Pi(\kappa, \mathcal{Y}(\kappa)) - \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\
&\times \Big(\int_0^1 \mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\psi'(\lambda)\mathrm{d}\lambda\Big) \\
&+ \frac{1}{\Gamma(\eta)} \int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}\mathcal{F}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))\mathrm{d}\lambda\Big)\Big| \\
&+ \Big(|\Pi(\kappa, \mathcal{X}(\kappa)) - \Pi(\kappa, \mathcal{Y}(\kappa))| + \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\
&\times \Big(\int_0^1 |\mathcal{G}(\lambda, \mathcal{X}(\lambda), \mathrm{B}\mathcal{X}(\lambda)) - \mathcal{G}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))|\, \psi'(\lambda)\mathrm{d}\lambda\Big) \\
&+ \frac{1}{\Gamma(\eta)} \int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1} |\mathcal{G}(\lambda, \mathcal{X}(\lambda), \mathrm{B}\mathcal{X}(\lambda)) - \mathcal{G}(\lambda, \mathcal{Y}(\lambda), \mathrm{B}\mathcal{Y}(\lambda))|\, \mathrm{d}\lambda\Big), \\
&\leq \frac{(\psi(1)-\psi(0))^\eta}{\Gamma(\eta+1)}\varsigma + \mathrm{L_G}\,|\mathcal{Y}(\kappa) - \mathcal{X}(\kappa)| + \frac{\Gamma(4-\eta)(\psi(1)-\psi(0))^{\eta-3}}{2}(\psi(\kappa)-\psi(0))^2 \\
&\times \Big(\int_0^1 (\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})\,|\mathcal{Y}(\lambda) - \mathcal{X}(\lambda)|\, \psi'(\lambda)\mathrm{d}\lambda\Big) \\
&+ \frac{1}{\Gamma(\eta)} \int_0^\kappa \psi'(\lambda)(\psi(\kappa)-\psi(\lambda))^{\eta-1}(\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})\,|\mathcal{X}(\lambda) - \mathcal{Y}(\lambda)|\, \mathrm{d}\lambda, \\
&\leq \frac{(\psi(1)-\psi(0))^\eta}{\Gamma(\eta+1)}\varsigma + \Big[\mathrm{L_G} + (\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})(\psi(1)-\psi(0))^\eta \Big[\frac{\Gamma(4-\eta)}{2} + \frac{1}{\Gamma(\eta+1)}\Big]\Big]\, \|\mathcal{X} - \mathcal{Y}\|\,.
\end{aligned}
$$

where

$$\alpha = \frac{(\psi(1)-\psi(0))^\eta}{\Gamma(\eta+1)}\varsigma.$$

Then

$$
\begin{aligned}
&|\mathcal{X}(\kappa) - \mathcal{Y}(\kappa)| \leq \alpha\varsigma + \Big[\mathrm{L_G} + (\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})(\psi(1)-\psi(0))^\eta \Big[\frac{\Gamma(4-\eta)}{2} + \frac{1}{\Gamma(\eta+1)}\Big]\Big]\, \|\mathcal{Y} - \mathcal{X}\| \\
&|\mathcal{X}(\kappa) - \mathcal{Y}(\kappa)| - \Big[\mathrm{L_G} + (\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})(\psi(1)-\psi(0))^\eta \Big[\frac{\Gamma(4-\eta)}{2} + \frac{1}{\Gamma(\eta+1)}\Big]\Big]\, \|\mathcal{Y} - \mathcal{X}\| \leq \alpha\varsigma, \\
&(1-\vartheta)\,|\mathcal{X}(\kappa) - \mathcal{Y}(\kappa)| \leq \alpha\varsigma. \qquad (13)
\end{aligned}
$$

Hence, from (13), we get

$$|\mathcal{X} - \mathcal{Y}| = \frac{\alpha\varsigma}{1-\vartheta}.$$

Hence, the problem (1) is U-H stable . □

5 Application

Example 15 *Consider the following NFIDEs using ψ-C-FDs of the problem:*

$$ {}^{C}D^{1/2}\mathcal{Y}(\kappa) - \frac{tan^{-1}|\mathcal{Y}(\kappa)|}{35} = \frac{\theta^3 + sin\,|\mathcal{Y}(\kappa)|}{45} + \int_0^{\kappa} \frac{1}{50}(\kappa^2+\lambda^2)\mathcal{Y}(\lambda)\mathrm{d}\lambda, \quad (14) $$

$$ \mathcal{Y}(0) = \mathcal{Y}'(0) = 0, \ {}^{C}\mathfrak{D}^{\eta-1;\psi}\mathcal{Y}(1) = 1, $$

Set

$$ \Pi(\theta, \mathcal{X}(\theta)) = \frac{tan^{-1}|\mathcal{X}(\theta)|}{35}, $$
$$ \mathcal{F}(\theta, \mathcal{X}(\theta), \mathrm{B}\mathcal{X}(\theta)) = \frac{\theta^3 + sin\,|\mathcal{X}(\theta)|}{45} $$
$$ \mathrm{B}\mathcal{X}(\theta)) = \int_1^{\kappa} \frac{1}{50}(\theta^2+\kappa^2)\mathcal{X}(\kappa)\mathrm{d}\kappa, $$

Hence, the conditions $(H_1 - H_4)$ hold, where $\eta = \frac{1}{2}$, $\mathrm{K}_1 = \mathrm{K}_2 = \frac{1}{45}$, $\mathrm{N} = \frac{1}{750}$ $\mathrm{L_G} = \frac{1}{35}$, By using Theorem 10, we determine that

$$ \left[\mathrm{L_G} + (\mathrm{K}_1 + \mathrm{K}_2\mathrm{N})(\psi(1) - \psi(0))^{\eta}\left[\frac{\Gamma(4-\eta)}{2} + \frac{1}{\Gamma(\eta+1)}\right]\right] < 0.047509 $$

In view of Theorem 10, the problem (14) *possesses a unique solution on the interval [0,1].*
Consequently, the solution $\mathcal{X}$ corresponding to problem (11) can be derived and is presented explicitly below. We obtain

$$ |\mathcal{Y} - \mathcal{X}| = \frac{\alpha\varsigma}{1-\vartheta} \leq 1.02634. $$

this demonstrates that the problem (14) is U-H stable.

6 Conclusion

We studied the uniquenes, existence and Ulam stability results for netural FIDE using ψ-C-FDs on the Banach space. Uniqueness results can be seen with the help of Banach's fixed point theorem. For existence results, Krasnoselskii's fixed point theorem is employed here. Lastly, we give an example to illustrate the validity of theoretical results. In the future, we intend to develop a numerical analysis for generalizing ψ-C-H-BVP with other types of FDs.

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