More on co-annihilating graph of commutative rings

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Abstract

In this paper, we characterize all commutative Artinian ring whose Co-annihilating graph $\mathcal{CA}_R$ is chordal or perfect.

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1 Introduction

The study of algebraic structures, using the properties of graphs, has become an exciting research topic in the last two decades, leading to many interesting results and questions. There are many papers on assigning a graph to a ring [3, 5]. The *comaximal graph* $\Gamma(R)$ with vertex set R and two vertices x and y are adjacent if and only if $Ra + Rb = R$. Let $\Gamma_2(R)$ be the subgraph of $\Gamma(R)$ induced by the non-units element of R [3]. Recently, Akbari et al [2] have introduced a graph namely *co-annihilating ideal* graph as follows. Let $A(R)$ be the set of all non-zero proper ideals of R. The co-annihilating ideal graph of R is defined as the graph $\mathcal{A}_{\mathcal{R}}$ with vertex set $A(R)$ and two distinct vertices I and J are adjacent whenever $Ann(I) \bigcap Ann(J) = \{0\}$. In [1], J. Amjadi et al have introduced and studied the properties of the co-annihilating graph of commutative ring. The *co-annihilating graph* of commutative ring R, denoted by $\mathcal{CA}_R$, is a simple graph with vertex set $\mathfrak{U}_R$ and two vertices x and y are adjacent whenever $Ann(x) \bigcap Ann(y) = (0)$. One can see that $\Gamma_2(R) \setminus J(R)$ is a subgraph of $\mathcal{CA}_R$, where $J(R)$ is the intersection of maximal ideals of R. Also an ideal I of R is an *essential ideal* if I has non-zero intersection with every other nonzero ideal of R.

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r -partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r *-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The *corona* of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ formed one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining the i^{th} vertex of G_1 is adjacent to every other vertex in the i^{th} copy of G_2. For a graph G, $X \subseteq V(G)$ is called a *clique* if the subgraph induced on X is complete. The number of vertices in the largest clique of graph G is called the *clique number* of G and it is denoted by $\omega(G)$. For a graph G, let $\chi(G)$ denote the *chromatic number* of G and defined the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect graph* G is a graph in which every induced subgraph is weakly perfect. A *chord* of a graph cycle C is an edge which is not in C but has both its end vertices in C. A graph G is *chordal* if every cycle of length at least 4 has chord.

Throughout this paper, we assume that R is Artinian ring with identity, $Z(R)$ is set of all zero divisor of R, $R^\times$ its group of units, $\mathfrak{U}_R$ be the set of all non-zero non-units of R, let $Nil(R)$, $Max(R)$ be the set of all nilpotent element, maximal ideal of R. In this paper, we characterize all commutative Artinian whose $\mathcal{CA}_R$ is chordal or perfect. We state without proof a few known results in the form of theorems, which will be used in the proofs of the main theorems.

Theorem 1. [4] *If R is an Artinian ring, then R is isomorphic to finite direct product of Artinian local rings.*

Theorem 2. [1, Observation 1.7] *Any non-zero nilpotent element of R is adjacent to only non-unit regular elements of $\mathcal{CA}_R$. In particular, if R has no non-unit regular element, then each non-zero nilpotent of R is an isolated vertex in $\mathcal{CA}_R$*

Theorem 3. [1, Corollary 2.2] *If $(R, \mathfrak{m})$ is an Artinian local ring, then $\mathcal{CA}_R$ is empty graph.*

Theorem 4. [1, Theorem 3.5] *Let R_1,$R_2, \ldots, R_n$ be Artinian local rings. If $R \cong R_1 \times R_2 \times \cdots \times R_n$, then $\omega(\mathcal{CA}_R) = \chi(\mathcal{CA}_R) = n$.*

Theorem 5. [1, Corollary 4.3], *Let R be an Artinian ring. Then $\mathcal{CA}_R$ is a complete bipartite graph if and only if R is isomorphic to the direct product of two fields.*

2 When $\mathcal{CA}_R$ is Perfect

In this section, we characterize all Artinian rings R, for which $\mathcal{CA}_R$ is perfect or chordal. We begin with the following theorems.

Theorem 6. *If R is an Artinian ring and $x \in Nil(R)^*$, then x is an isolated vertex in $\mathcal{CA}_R$.*

Proof. Assume that x is a non-zero nilpotent element of R. We claim that $Ann(x)$ is an essential ideal of R. Suppose to the contrary, there exists an ideal I such that $I \cap Ann(x) = (0)$. Thus $yx \neq 0$, for every $y \in I$. Obviously, $yx \in I$ and so $(yx)x = yx^2 \neq 0$. By continuing this procedure, $yx^n \neq 0$, for every positive integer n, a contradiction. Hence $Ann(x)$ is an essential ideal of R and so $Ann(x) \cap Ann(l) \neq (0)$, for every $l \in Z(R)^*$. Therefore x is an isolated vertex of $\mathcal{CA}_R$. □

Theorem 7. *Let R be an Artinian ring. Then the following statements are equivalent.*

(i) $|Max(R)| = 1$

(ii) $\mathcal{CA}_R = \overline{K_n}$ *, where* $n = |\mathfrak{m}^*|$.

Proof. We assume that R is an Artinian ring and $|Max(R)| = 1$. Then R is Artinian local ring and $\mathfrak{m} = Nil(R)$. By theorem 6, we have $\mathcal{CA}_R = \overline{K_n}$, where $n = |\mathfrak{m}^*|$. Suppose $|Max(R)| \geq 2$. By theorem 4, $\mathcal{CA}_R$ has a clique of size greater than 2. So $|Max(R)| = 1$. □

Theorem 8. *Let R be an Artinian ring. Then $\mathcal{CA}_R$ is chordal if and only if one of the following ring:*
(i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
(ii) $\mathbb{Z}_2 \times F$
(iii) *R is local*

Proof. Assume that $\mathcal{CA}_R$ is a chordal graph. By the assumption on R, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is local rings. If $n \geq 4$, then $(0,1,1,0,1,\ldots,1) - (1,0,0,1,1,\ldots,1) - (1,1,1,0,1,\ldots,1) - (1,0,1,1,\ldots,1) - (0,1,1,0,1,\ldots,1)$, is an induced cycle of length 4 and $\mathcal{CA}_R$ is not chordal, a contradiction. Hence $n \leq 3$.

Case 1. $n = 3$. If $|R_3| \geq 3$, then $(1,0,0) - (0,1,1) - (1,1,0) - (0,1,u) - (1,0,0)$ is an induced cycle of length 4 in $\mathcal{CA}_R$, for some $1 \neq u \in R_3^{\times}$, a contradiction. Hence $|R_i|$=2 for all i and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. $n = 2$. If $\mathfrak{m}_2 \neq (0)$, then $(1,0) - (0,1) - (1,x) - (0,u) - (1,0)$ is an induced cycle of length 4 in $\mathcal{CA}_R$ for some $x \in \mathfrak{m}_2^*$ and $1 \neq u \in R_2^{\times}$, a contradiction. Hence R_1 and R_2 are fields and by theorem , $\mathcal{CA}_R \cong K_{|R_1|-1,|R_2|-1}$. Since $\mathcal{CA}_R$ is chordal, so $R \cong \mathbb{Z}_2 \times F$, where F is a field. Finally, if $n = 1$, then by theorem 4, $\mathcal{CA}_R \cong \overline{K_q}$, where $q = |\mathfrak{m}^*|$.

Conversely, suppose $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{CA}_R \cong K_3 \circ K_1$. If $R \cong \mathbb{Z}_2 \times F$, then $CA_R \cong K_{1,q}$, where $q = |F| - 1$. □

The following theorem, we characterize all Artinian rings R whose $\mathcal{CA}_R$ is perfect. We give some following much needed reference theorems.

Theorem 9. [8] *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

Theorem 10. [8] *Let G be a graph. Then the following statements hold:*
(i) *G is a perfect if and only if $\overline{G}$ is a perfect graph.*
(ii) *If G is a complete bipartite graph, then G is a perfect graph.*
(iii) *Every chordal graph is perfect.*

Theorem 11. *Let R be an Artinian rings. Then $\mathcal{CA}_R$ is perfect if and only if $|Max(R)| \leq 4$.*

Proof. Assume that $\mathcal{CA}_R$ is perfect. Since R is Artinian ring, there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local rings, for every $1 \leq i \leq n$. Suppose $|Max(R)| \geq 5$, then there exists an induced odd cycle $(x_1, x_2, u_3, u_4, u_5, u_6, \cdots, u_n)$ - $(u_1, u_2, x_3, u_4, x_5, u_6, \cdots, u_n)$ - $(x_1, u_2, u_3, x_4, u_5, u_6, \cdots, u_n)$ - $(u_1, x_2, u_3, u_4, x_5, u_6, \cdots, u_n)$ - $(u_1, u_2, x_3, x_4, u_5, u_6, \cdots, u_n)$ - $(x_1, x_2, u_3, u_4, u_5, u_6, \cdots, u_n)$, where $x_i \in Z(R_i)$, $u_i \in R_i^{\times}$, for every $1 \leq i \leq n$. Thus $|Max(R)| \leq 4$.

Conversely, Suppose that $|Max(R)| \leq 4$. We prove neither $\mathcal{CA}_R$ nor $\overline{\mathcal{CA}_R}$ contains an induced odd cycle of length at least 5.

Claim A. We shows that $\mathcal{CA}_R$ contains no induced odd cycle of length at least 5. Suppose that $\mathcal{CA}_R$ contains odd cycle of length at least 5.

Case 1. First, we shows that $\mathcal{CA}_R$ contains no induced odd cycle of length at least 5. Suppose $|Max(R)| \geq 5$, then there exists an induced odd cycle $(x_1, x_2, u_3, u_4, u_5, u_6, \cdots, u_n)$ - $(u_1, u_2, x_3, u_4, x_5, u_6, \cdots, u_n)$ - $(x_1, u_2, u_3, x_4, u_5, u_6, \cdots, u_n)$ - $(u_1, x_2, u_3, u_4, x_5, u_6, \cdots, u_n)$ - $(u_1, u_2, x_3, x_4, u_5, u_6, \cdots, u_n)$ - $(x_1, x_2, u_3, u_4, u_5, u_6, \cdots, u_n)$, where $x_i \in Z(R_i)$, $u_i \in R_i^{\times}$, for every $1 \leq i \leq n$. Thus $n \leq 4$.

If $n = 4$, then we consider the partition for $V(\mathcal{CA}_R)$. $A_1 = \{(x_1, u_2, u_3, u_4)\}$, $A_2 = \{(u_1, x_2, u_3, u_4)\}$, $A_3 = \{(u_1, u_2, x_3, u_4)\}$, $A_4 = \{(u_1, u_2, u_3, x_4)\}$, $B_1 = \{(x_1, u_2, u_3, x_4)\}$, $B_2 = \{(x_1, u_2, x_3, u_4)\}$, $B_3 = \{(x_1, x_2, u_3, u_4)\}$, $B_4 = \{(u_1, u_2, x_3, x_4)\}$, $B_5 = \{(u_1, x_2, x_3, u_4)\}$, $B_6 = \{(u_1, x_2, u_3, x_4)\}$, $C_1 = \{(u_1, x_2, x_3, x_4)\}$, $C_2 = \{(x_1, u_2, x_3, x_4)\}$, $C_3 = \{(x_1, x_2, u_3, x_4)\}$,

$C_4 = \{(x_1, x_2, x_3, u_4)\}$, $D = \{(x_1, x_2, x_3, x_4)\}$, where $x_i \in Z(R_i)$, $u_i \in R_i^{\times}$, for every $1 \le i \le n$.

If we put $A = \cup_{i=1}^{4} A_i$, $B = \cup_{i=1}^{6} B_i$ and $C = \cup_{i=1}^{4} C_i$, then one may check that A, B, C and D is a partition of $V(\mathcal{CA}_R)$. Suppose, we assume to contrary, $a_1 - a_2 - \cdots - a_\ell - a_1$ is an induced odd cycle of length at least 5. By theorem 6, every vertex in D is isolated vertex in $\mathcal{CA}_R$ and thus $\{a_1 - a_2 - \cdots - a_\ell\} \cap D = \emptyset$. We shows that $\{a_1 - a_2 - \cdots - a_\ell\} \cap C_1 = \emptyset$. Suppose $a_j \in \{a_1 - a_2 - \cdots - a_\ell\} \cap C_1$, for some $1 \le j \le \ell$. Without loss of generality, assume that $a_1 \in C_1$. Since every vertex of C_1 is adjacent only to vertices of A_1, $a_2, a_\ell \in A_1$. Hence there exists a vertex of $\mathcal{CA}_R$ is adjacent to a_2 if and only if it is adjacent to a_ℓ. Therefore $\{a_1 - a_2 - \cdots - a_\ell\} \cap C_1 = \emptyset$. Similarly, we concluded that $\{a_1 - a_2 - \cdots - a_\ell\} \cap C_i = \emptyset$, for every $2 \le i \le 4$. Hence $\{a_1 - a_2 - \cdots - a_\ell\} \cap C = \emptyset$.

Now, we show that $\{a_1 - a_2 - \cdots - a_\ell\} \cap B_1 = \emptyset$. If $a_1 \in B_1$, then a_1 is adjacent to only to the vertices of $B_5 \cup A_3 \cup A_2$ and $\{a_2, a_\ell\} \subseteq B_5 \cup A_3 \cup A_2$. If $a_2 \in B_5$, then it is easy to see that a_3 is adjacent to a_ℓ in cycle C_ℓ. Thus $a_2 \notin B_5$, $a_\ell \notin B_5$ and $a_2, a_\ell \subseteq \{A_3 \cup A_2\}$. Clearly, $\mathcal{CA}_R[A_3 \cup A_2]$ is a complete bipartite graph. Hence we concluded that $\{a_3, a_\ell\} \subseteq A_3$ or $\{a_3, a_\ell\} \subseteq A_2$. With no loss of generality, we assume that $\{a_3, a_\ell\} \subseteq A_3$. This implies that a_3 is adjacent to a_2 and a_ℓ, a contradiction and that $\{a_1 - a_2 - \cdots - a_\ell\} \cap B_1 = \emptyset$. So $\{a_1 - a_2 - \cdots - a_\ell\} \cap B_i = \emptyset$, for every $2 \le i \le 6$ and $\{a_1 - a_2 - \cdots - a_\ell\} \subseteq A$. It is clearly that $\mathcal{CA}_R[A]$ is a complete 4-partite graph with partite sets A_i for $1 \le i \le 4$, a contradiction. Therefore, if $n = 4$, then $\mathcal{CA}_R$ contains no induced odd cycle of length at least 5.

Case 2. $n = 3$. In this case, $R \cong R_1 \times R_2 \times R_3$. let $x_i \in Z(R_i)$ and $u_i \in R_i^{\times}$, for every $1 \le i \le 3$. We consider the following partitions for $V(\mathcal{CA}_R)$: $A_1 = \{(x_1, x_2, x_3) \backslash (0, 0, 0)\}$, $A_2 = \{(u_1, x_2, x_3)\}$, $A_3 = \{(x_1, u_2, x_3)\}$, $A_4 = \{(x_1, x_2, u_3)\}$, $B_1 = \{(u_1, u_2, x_3)\}$, $B_2 = \{(u_1, x_2, u_3)\}$, $B_3 = \{(x_1, u_2, u_3)\}$. Let $A = \cup_{i=1}^{4} A_i$ and $B = \cup_{i=1}^{3} B_i$. Assume to the contrary, $a_1 - a_2 - \cdots - a_k - a_1$ is an induced odd cycle of length 5 in $\mathcal{CA}_R$. By theorem 6, every vertex of A_1 is an isolated vertex in $\mathcal{CA}_R$. We show that $\{a_1, a_2, \cdots, a_k\} \cap A_2 = \emptyset$. If $a_1 \in A_2$, then a_1 is adjacent to only to the vertices of B_3. So the vertex a_3 is adjacent to both a_2 and a_k, a contradiction. Thus $\{a_1, a_2, \cdots, a_k\} \cap A_2 = \emptyset$. Similarly, $\{a_1, a_2, \cdots, a_k\} \cap A_3 = \emptyset$ and $\{a_1, a_2, \cdots, a_k\} \cap A_4 = \emptyset$. It is clearly that $\mathcal{CA}_R[B]$ is a complete 3-partite graph with partite sets B_i for $1 \le i \le 3$, a contradiction. Hence if $n = 3$, then $\mathcal{CA}_R$ contains no induced odd cycle of length at least 5.

Case 3. $n = 2$. In this case, $R \cong R_1 \times R_2$ and $|Max(R)| = 2$. If $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, then by Theorem 5 and 10, we have $\mathcal{CA}_R$ is complete bipartite and perfect graph. Suppose that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$. Let $A = \{(x, v) : x \in Z(R_1), v \in R_2^{\times})\}$, $B = \{(u, y) : y \in Z(R_2), u \in R_1^{\times})\}$ and $C = \{(x, y) \setminus (0, 0) : x \in Z(R_1), y \in Z(R_2)\}$. Clearly, $\mathcal{CA}_R[C]$ is isolated vertices of $\mathcal{CA}_R$. Also, it is easy to see that $\mathcal{CA}_R[A \cup B]$ is complete bipartite. Thus if $n = 2$, then $\mathcal{CA}_R$ contains no induced odd cycle of length at least 5.

Case 4. Finally, If $n = 1$, then $|Max(R)| = 1$ and by theorem 6, deduce that $\mathcal{CA}_R \cong \overline{K_n}$.

Claim B. $\overline{\mathcal{CA}_R}$ contains no induced odd cycle of length at least 5. Suppose that $\overline{\mathcal{CA}_R}$ contains odd cycle of length at least 5. Suppose $n \ge 5$. Since $\mathcal{CA}_R$ is not perfect. By Theorem 10, $\overline{\mathcal{CA}_R}$ is not perfect. Thus $n \le 4$.

Case 1. Let $n = 4$. We consider the partitions of above claim A and Case 1.

Suppose, we assume to contrary, $a_1 - a_2 - \cdots - a_\ell - a_1$ is an induced odd cycle of length at least 5. By theorem6, every vertex in D is isolated vertex in $\mathcal{CA}_R$ and thus $\overline{\mathcal{CA}_R[D]}$ is complete subgraph, every vertex D is adjacent to for all other vertices of $\overline{\mathcal{CA}_R}$. Hence $\{a_1 - a_2 - \cdots - a_\ell\} \cap D = \emptyset$. We shows that $\{a_1 - a_2 - \cdots - a_\ell\} \cap A_1 = \emptyset$. Suppose $a_j \in \{a_1 - a_2 - \cdots a_\ell\} \cap A_1$, for some $1 \le j \le \ell$. Without loss of generality, if $a_1 \in A_1$, then the first components of a_2 and a_ℓ must be in $Z(R_1)$. So that a_2

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adjacent to a_ℓ, a contradiction. Therefore $\{a_1 - a_2 - \cdots - a_\ell\} \cap A_1 = \emptyset$. Similarly, we concluded that $\{a_1 - a_2 - \cdots - a_\ell\} \cap A_i = \emptyset$, for every $2 \leq i \leq 4$. Hence $\{a_1 - a_2 - \cdots - a_\ell\} \cap A = \emptyset$.

Now, we show that $\{a_1 - a_2 - \cdots - a_\ell\} \cap C_1 = \emptyset$. Since $\overline{\mathcal{CA}_R[C_i]}$ is complete graph of $\overline{\mathcal{CA}_R}$ for every, $1 \leq i \leq 4$. If $a_1 \in C_1$, then a_1 is adjacent to only to the vertices of $B_i \cup D$. Also, it is easy to see that every vertices in C_1 is adjacent to B and D, a contradiction. Therefore, $\{a_1 - a_2 - \cdots - a_\ell\} \cap C_1 = \emptyset$. So $\{a_1 - a_2 - \cdots - a_\ell\} \cap C_i = \emptyset$, for every $2 \leq i \leq 4$ and $\{a_1 - a_2 - \cdots - a_\ell\} \subseteq B$. It is clearly that $|\{a_1 - a_2 - \cdots - a_\ell\} \cap B_i| \leq 1$, where $1 \leq i \leq 6$. Then $\overline{\mathcal{CA}_R[B]} \cong K_{m,n} - E(G)$, where $E(G) = (B_2, B_4)$ with partite sets $\{B_1, B_2, B_5\}$ and $\{B_3, B_4, B_6\}$. Therefore, if $n = 4$, then $\overline{\mathcal{CA}_R}$ contains no induced odd cycle of length at least 5.

Case 2. Let $n = 3$, We consider the partitions of above claim A and case 2. Let $A = \cup_{i=1}^{4} A_i$ and $B = \cup_{i=2}^{3} B_i$. Assume to the contrary, $a_1 - a_2 - \cdots - a_k - a_1$ is an induced odd cycle of length 5 in $\overline{\mathcal{CA}_R}$. By theorem 6, every vertex of A_1 is adjacent to for all other vertex of $\overline{\mathcal{CA}_R}$. We show that $\{a_1, a_2, \cdots, a_k\} \cap B_1 = \emptyset$. If $a_1 \in B_1$, then the last components a_2 and a_k must be in $Z(R)$. Hence that a_2 is adjacent to a_k. Thus $\{a_1, a_2, \cdots, a_k\} \cap B_1 = \emptyset$. Similarly, $\{a_1, a_2, \cdots, a_k\} \cap B_2 = \emptyset$ and $\{a_1, a_2, \cdots, a_k\} \cap B_3 = \emptyset$. It is clearly that $\overline{\mathcal{CA}_R[A]}$ is a induced complete graph. Hence if $n = 3$, then $\mathcal{CA}_R$ contains no induced odd cycle of length at least 5.

Case 3. $n = 2$. In this case, $R \cong R_1 \times R_2$ and $|Max(R)| = 2$. If $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, then by Theorem 5 and 10, we have $\overline{\mathcal{CA}_R}$ is disjoint union of complete graph and perfect graph. Suppose that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$. Let $A = \{(x, v) : x \in Z(R_1), v \in R_2^\times\}$, $B = \{(u, y) : y \in Z(R_2), u \in R_1^\times)\}$ and $C = \{(x, y) \setminus (0, 0) : x \in Z(R_1), y \in Z(R_2)\}$. Clearly, $\mathcal{CA}_R[C]$ is induced complete subgraph $\overline{\mathcal{CA}_R}$. Also, it is easy to see that $\mathcal{CA}_R[A \cup B]$ is disjoint union of complete graph. Hence if $n = 2$, then $\overline{\mathcal{CA}_R}$ contains no induced odd cycle of length at least 5. Finally, if $n = 1$, then $\overline{\mathcal{CA}_R}$ is complete. Therefore Claim A, Claim B and Theorem 9, we have $\mathcal{CA}_R$ is a perfect graph.

□

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