A note on completely regular 2-duo ordered Γ-semihypergroups by relative ordered quasi-Γ-hyperideals and relative ordered bi-Γ-hyperideals

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ABSTRACT. In this paper, we show that if H is a duo ordered Γ-semihypergroup, then the sets $L(S)$ and $R(S)$ coincide, where the sets $L(A)$ and $R(A)$ are called the relative left Γ-hyperideal and the relative right Γ-hyperideal of H generated by A, in that order. We prove that if H is a duo ordered Γ-semihypergroup, then H is an n-duo ordered Γ-semihypergroup where $n \geq 2$. We obtain necessary and sufficient conditions for relative completely regular 2-duo ordered Γ-semihypergroup for all relative $(0,2)$-Γ-hyperideal of H and for all relative $(2,0)$-Γ-hyperideal of H.

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1. INTRODUCTION

The notion of bi-ideal (i.e., $(1,1)$-ideal) was introduced by Good and Hughes [36]. Lajos [37] generalized this notion of bi-ideal with introducing (m,n)-ideal. The concept of quasi-deals was introduced by Steinfeld [29] both in rings and in semigroups. For properties of quasi-ideals, one can refer [30]. Christoph [16] and Iseki [21], [22], [23] studied quasi-ideals in semirings whereas Weinert [19] studied quasi-ideals in rings. Also, the theory of quasi-ideals in ternary semigroups was studied by Dixit et al. [40]. The concept of ordered quasi-ideals in ordered semigroups was introduced by Kehayopulu [27].

Davvaz et al. [12, 13, 18, 38] introduced the notion of Γ-semihypergroup as a generalization of semigroup, semihypergroup and Γ-semigroup. The notion of ordered Γ-semihypergroup was introduced by Kondo and Lekkoksung [20]. They studied bi-hyperideals, quasi-hyperideals, left hyperideals and right hyperideals in ordered Γ-semihypergroups and characterized intra-regular ordered Γ-semihypergroups by their bi-hyperideals and quasi-hyperideals. Ansari et al [39] studied regular and intra-regular Γ–semihypergroups.

The theory of T-ideal (or relative ideal) in semigroup S (resp. left, right relative ideals), where $T \subseteq S$, was initiated by Wallace [8], [9]. Khan et al. [26] introduced relative ideals in ordered semigroups. Thereafter, Basar et al [1], [2], [3], [4] [5], [6], [7] studied relative ideals in different algebraic structures.

The concept of hyperstructures was given by Marty [17]. Hila et al. [24] studied quasi-hyperideals in semihypergroups. Heideri et al . [15] studied ordered hyperstructures. For useful review of various algebraic hyperstructures and their applications in different fields, one can refer [14], [31], [32], [33], [34].

Motivations for the investigation of hyperstructures arises mainly from inside the domains of computer science, mathematics, biological inheritance relation, theoretical physics, cryptography, physical phenomenon as the nuclear fission, chemical reactions and redox reactions and other fields of study from social sciences. This wide range of its versatile applications in various domains led ample scope for the expansion and exploration of hyperstructures in current decades, such as H_v-structures. Hyperideal theory is crucial not only for the intrinsic vital interest and purity of its logical algebraic structures but since it is a necessary prerequisite model in many branches of mathematics and its applications in various fields of sciences as well as social sciences. Many related results have been obtained in Γ-semihypergroup. However, the very fundamental characterizations of Γ-hyperideals in different important classes of Γ-semihypergroups remained so far untouched. This gives us enough impetus to study these classes of hyperstructures to further the current research in terms of relative hyperideal theory.

Kehayopulu [20] introduced the notion of duo ordered semigroups. In this paper, we explore characterizations of completely regular 2-duo ordered Γ-semihypergroups by relative Γ-hyperideals. We show that if H is a duo ordered Γ-semihypergroup, then the sets $L(S)$ and $R(S)$ coincide. Also, we prove that if H is a duo ordered Γ-semihypergroup, then H is an n-duo ordered Γ-semihypergroup where $n \geq 2$. We shed light on necessary and sufficient conditions for relative completely regular 2-duo ordered Γ-semihypergroup studied by Luangchaisri et al [35] and Jantanan et al [41] for all relative $(0,2)$-Γ-hyperideal of H and for all relative $(2,0)$-Γ-hyperideal of H.

2. Preliminaries

In this section, we note the preliminaries which will be used throughout this article following [5]. A hyperstructure H is a nonvoid set equipped with an hyperoperation "$\circ$" on H defined as follows:

$$\circ : H \times H \to \mathscr{P}^*(H) \mid (x,y) \to (x \circ y)$$

and an operation "$*$" on $\mathscr{P}^*(H)$ defined as follows:

$$* : \mathscr{P}^*(H) \times \mathscr{P}^*(H) \to \mathscr{P}^*(H) \mid (X,Y) \to X * Y$$

such that

$$X * Y = \bigcup_{(x,y)\in X\times Y} (x \circ y)$$

for any $X,Y \in \mathscr{P}^*(H)$, where $\mathscr{P}^*(H)$ denotes the nonempty subsets of H. A hyperoperation "$\circ$" on H gives rise to an operation "$*$" on $\mathscr{P}^*(H)$. Conversely, an operation "$*$" on $\mathscr{P}^*(H)$ gives rise to a hyperoperation "$\circ$" on H, defined as follows: $x \circ y = \{x\} * \{y\}$. Therefore, a hypersemigroup $(H,\circ,*)$ can be identified by $(H,\circ)$ because of the interdependency of the operation "$*$" and the hyperoperation "$\circ$". Clearly, we have $X \subseteq Y \Rightarrow X * D \subseteq Y * D, D * X \subseteq D * Y$ for any $X,Y,D \in \mathscr{P}^*(H)$ and $H * H \subseteq H$. For a subset X of an hypersemigroup H, we define by $(X]$ the subset of H as follows:

$$(X] = \{s \in H :| s \leq x \text{ for some x} \in \text{X}\}.$$

If "$\leq$" is an order relation on a hypersemigroup H, we define the order relation "$\preceq$" on $\mathscr{P}^*(H)$ as follows:

$$\preceq := \{(X,Y) \mid \forall x \in X \; \exists \, y \in Y \text{ such that x} \leq \text{y}\}.$$

Therefore, for $X,Y \in \mathscr{P}^*(H)$, we denote $X \preceq Y$ if for every $x \in X$, there exists $y \in Y$ such that $x \leq y$. This is indeed, a reflexive and transitive relation on $\mathscr{P}^*(H)$.

A hyperstructure $(H,\circ)$ is called a semihypergroup if for all $x,y,z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, i. e.,

$$\bigcup_{m\in x\circ y} m \circ z = \bigcup_{n\in y\circ z} x \circ n.$$

A nonempty subset A of a semihypergroup $(H,\circ)$ is called a subsemihypergroup of H if $A * A \subseteq A$. An semihypergroup $(H,\circ)$ equipped with a partial order "$\leq$" on H that is compatible with semihypergroup operation "$\preceq$" such that for all $x,y,z \in H$,

$$x \leq y \Rightarrow z \circ x \preceq z \circ y \text{ and x} \circ \text{z} \preceq \text{y} \circ \text{z},$$

ia called an ordered semihypergroup. Let $(H,\circ,\leq)$ be an ordered semihypergroup, $S \subseteq H$ and let X,Y,Z be nonempty subsets of S, then we easily have the following [5]:

(i) If $x \in X * Y$, then $x \in x' \circ y$ for some $x' \in X$, $y \in Y$.
(ii) If $x \in X$, $y \in Y$, then $x \circ y \subseteq X * Y$.
(iii) $X \subseteq (X]_S$;
(iv) If $X \subseteq Y$, then $(X]_S \subseteq (Y]_S$;
(v) $(X]_S * (Y]_S \subseteq (X * Y]_S$;
(vi) $((X]_S * (Y]_S]_S = (X * Y]_S$;
(vii) For every left (resp. right) S-hyperideal I of S, $(I]_S = I$.
(viii) $(X \cap Y) \subseteq (X]_S \cap (Y]_S$.
(ix) $X \cup Y = (X]_S \cup (Y]_S$.
(x) $((X_S]_S]_S = (X]_S$.
(xi) $X * (Y \cap Z) \subseteq X * Y \cap X * Z$.
(xii) $X * (Y \cup Z) = X * Y \cup X * Z$.
(xiii) If A,B are S-hyperideals of H such that $A \cap B \neq \varnothing$, then $(A * B]_S$, $A \cap B$ are S-hyperideals of H.
(xiv) If I is a sub-hypersemigroup of H, then $(I * s * I]_I$ is a I-hyperideal of H for each $s \in H$.
(xv) If H is a hypersemigroup and $X,Y,Z \in \mathscr{P}^\star(H)$, then we define $(X * Y) * Z = X * (Y * Z)$; and if H is an ordered hypersemigroup and $A,B \subseteq H$, then
(xvi) $(\bigcup_{i\in I} A_i) * B = \bigcup_{i\in I} (A_i * B)$.
(xvii) $B * (\bigcup_{i\in I} A_i) = \bigcup_{i\in I} (B * A_i)$.

Throughout this paper, H will denote an ordered Γ-semihypergroup unless otherwise stated.

Definition 2.1. *Suppose that $(H,\circ,\leq)$ is an ordered Γ-semihypergroup and $S\subseteq H$. Then, a nonempty subset I of H is called a right (resp., left) relative Γ-hyperideal of H if*

(i) $I\circ\Gamma\circ S\subseteq I$*(resp.,* $S\circ\Gamma\circ I\subseteq I$*); and*
(ii) *if* $x\in I$ *and* $S\ni y\leq x$*, then* $y\in I$*, i. e., if* $(I]_S=I$.

A subset of H which is both a right and left relative Γ-hyperideal of H is called a relative Γ-hyperideal of H. We see that $I\circ\Gamma\circ S\subseteq I(resp.,\ S\circ\Gamma\circ I\subseteq I)\iff x\circ s\subseteq I(resp.,\ s\circ\Gamma\circ x\subseteq I)$ for every $x\in I$, and every $s\in S$. Clearly, every right(resp., left) relative Γ-hyperideal of an ordered Γ-semihypergroup H is a sub-Γ-semihypergroup of H.

Definition 2.2. *Suppose that $(H,\circ,\leq)$ is an ordered Γ-semihypergroup, and let $S\subseteq H$. A nonempty subset Q of H is called a relative quasi Γ-hyperideal of H if*

(i) $(S\circ\Gamma\circ Q]_S\cap(Q\circ\Gamma\circ S]_S\subseteq Q$*; and*
(ii) $p\in Q, S\ni q\leq p\Rightarrow q\in Q$*, i. e.,* $(Q]_S=Q$.

Mahboob et al [10, 11] studied (m,n)-quasi-hyperideals and (m,n)-hyperideals in ordered semihypergroups.

Definition 2.3. *Suppose that $(H,\circ,\leq)$ is an ordered Γ-semihypergroup, $S\subseteq H$ and m,n are nonnegative integers. A nonempty subset Q of H is called a relative (m,n)-quasi Γ-hyperideal of H if*

(i) $(S^m\circ\Gamma\circ Q]_S\cap(Q\circ\Gamma\circ S^n]_S\subseteq Q$*; and*
(ii) $p\in Q, S\ni q\leq p\Rightarrow q\in Q$*, i. e.,* $(Q]_S=Q$.

Definition 2.4. *Let $(H,\circ,\leq)$ be an ordered Γ-semihypergroup and let S be any nonempty subset of H. Then, a sub-Γ-semihypergroup B of H is said to be a relative bi-Γ-hyperideal of H if*

(i) $B\circ\Gamma\circ S\circ\Gamma\circ B\subseteq B$*; and*
ii) *for all* $t\in B$, $S\ni g\leq t\Rightarrow g\in B$.

Definition 2.5. *An ordered Γ-semihypergroup H is called relative regular (resp. relative left regular, relative right regular) if for every $s\in S\subseteq H$, $s\in(s\circ\Gamma\circ S\circ\Gamma\circ s]_S$(resp. $s\in(S\circ\Gamma\circ s^2]_S, s\in(s^2\circ\Gamma\circ S]_S$).*

Definition 2.6. *An ordered Γ-semihypergroup H is called completely relative regular if it is both relative right regular and relative left regular.*

Ardekani and Davvaz [25] defined the following notion for ordered semihypergroup. We define it in ordered Γ-semihypergroup in terms of relative ordered Γ-hyperideals as follows:

Definition 2.7. *An ordered Γ-semihypergroup H is called right (resp. left) relative duo if the right (resp. left) relative Γ-hyperideals of H are two-sided. Also, H is called relative duo if it is both right relative duo as well as left relative duo.*

Definition 2.8. *Suppose that H is an ordered Γ-semihypergroup and let n be a positive integer. Then H is said to be an n-duo ordered Γ-semihypergroup if it satisfies the following conditions:*

(1): *every relative $(n,0)$-Γ-hyperideal of H is a relative $(0,n)$-Γ-hyperideal of H; and*
(2): *every relative $(0,n)$-Γ-hyperideal of H is a relative $(n,0)$-Γ-hyperideal of H.*

Suppose that H is an ordered Γ-semihypergroup and A is any non-empty subset of H. Then the relative (m,n)-Γ-hyperideal $[A]_{m,n}$ is called the relative (m,n)-Γ-hyperideal of H generated by A. Similarly, $[A]_{m,0}$ and $[A]_{0,n}$ are called the relative $(m,0)$ -Γ-hyperideal and the relative $(0,n)$-Γ-hyperideal of H generated by A, respectively. Thus we have the following:

$$[A]_{m,n}=(\bigcup_{i=1}^{m+n}A^i\cup A^m\circ\Gamma\circ H\circ\Gamma\circ A^n].$$

Furthermore, if $A = \{a\}$, we denote $[\{a\}]_{m,n}$ by $[a]_{m,n}$. It is to be noted that if H is a 2-duo ordered Γ-semihypergroup, then $[a]_{0,2} = [a]_{2,0}$ for all $a \in H$.

Suppose that H is an ordered Γ-semihypergroup and A is a non-empty subset of H. Then the sets $L(A)$ and $R(A)$ are called the relative left Γ-hyperideal and the relative right Γ-hyperideal of H generated by A, in that order. One can clearly obtain $L(A) = (A \cup S \circ \Gamma \circ A]_S$ and $R(A) = (A \cup A \circ \Gamma \circ S]_S$ for $S \subseteq H$.

3. Main Results

Now, in this penultimate section, we start with the following:

Lemma 3.1. *Suppose that H is an ordered Γ-semihypergroup. Then the following statements are equivalent for $S \subseteq H$:*

(1): *H is completely relative regular;*
(2): *$A \subseteq (A^2 \circ \Gamma \circ S \circ \Gamma \circ A^2)$ for all $A \subseteq H$; and*
(3): *$s \in (s^2 \circ \Gamma \circ H \circ \Gamma \circ s^2)$ for all $s \in S \subseteq H$.*

Lemma 3.2. *Suppose that H is an ordered Γ-semihypergroup and A is a non-empty subset of H. If H is a duo ordered Γ-semihypergroup, then the sets $L(S)$ and $R(S)$ coincide.*

Proof. Suppose that H is a duo ordered Γ-semihypergroup and assume that $\varnothing \neq S \subseteq H$. Also, let $A \subseteq H$. Let $s \in L(S) = (S \cup A \circ \Gamma \circ S]_A = (S] \cup (A \circ \Gamma \circ S]_A$. Then, we receive $s \in (S]$ or $s \in (A \circ \Gamma \circ S]_A$. If $s \in (S]$, then $s \in (S \cup S \circ \Gamma \circ A]_A$. If $s \in (A \circ \Gamma \circ S]_A$, then $s \in (A \circ \Gamma \circ (S \cup S \circ \Gamma \circ A]_A]_A$. As $(S \cup S \circ \Gamma \circ A]_A$ is a relative right Γ-hyperideal of H, where H is a duo ordered Γ-semihypergroup, $(S \cup S \circ \Gamma \circ A]_A$ is a relative duo left Γ-hyperideal of H. This follows that $A \circ \Gamma \circ (S \cup S \circ \Gamma \circ A]_A \subseteq (S \cup S \circ \Gamma \circ A]_A$. Therefore, we thus receive the following:

$$s \in (A \circ \Gamma \circ (S \cup S \circ \Gamma \circ A]_A]_A \subseteq ((S \cup S \circ \Gamma \circ A]_A]_A = (S \cup S \circ \Gamma \circ A]_A.$$

So, $s \in (S \cup S \circ \Gamma \circ A]_A = R(S)$. This implies that $L(S) \subseteq R(S)$. The condition that $R(S) \subseteq L(S)$ can be proved in a similar fashion. Hence $L(S) = R(S)$.

□

Theorem 3.3. *Suppose that H is an ordered Γ-semihypergroup. If H is a duo ordered Γ-semihypergroup, then H is an n-duo ordered Γ-semihypergroup where $n \geq 2$.*

Proof. Suppose that H is a duo-ordered Γ-semihypergroup. Let I be a relative $(n,0)$-Γ-hyperideal of H. We will prove that I is a relative $(0,n)$-Γ-hyperideal of H. Let us observe the following:

$$\begin{aligned}
A \circ \Gamma \circ I^n &\subseteq (I^n \cup A \circ \Gamma \circ I^n) \\
&= ((I \cup A \circ \Gamma \circ I) \circ \Gamma \circ I^{n-1}]_A \\
&\subseteq ((I \cup A \circ \Gamma \circ I]_A \circ \Gamma \circ (I^{n-1}]_A]_A \\
&= (L(I) \circ \Gamma \circ (I^{n-1})]_A]_A \\
&= (R(I) \circ \Gamma \circ (I^{n-1})]_A]_A \\
&= ((I \cup I \circ \Gamma \circ A]_A \circ \Gamma \circ (I^{n-1}]_A]_A \\
&= (I^n \cup I \circ \Gamma \circ A \circ I^{n-1}]_A \\
&\subseteq (I \cup I \circ \Gamma \circ A \circ \Gamma \circ I^{n-1}]_A.
\end{aligned}$$

In the similar fashion, we find the following:

$$A \circ \Gamma \circ I^{n-1} \subseteq (I \cup I \circ \Gamma \circ A \circ \Gamma \circ I^{n-2}]_A.$$

Therefore, we obtain the following:

$$\begin{aligned} A\circ\Gamma\circ I^n &\subseteq (I\cup I\circ\Gamma\circ(A\circ\Gamma\circ I^{n-1})]_A \\ &\subseteq (I\cup(I]_A\circ\Gamma\circ(I\cup I\circ\Gamma\circ A\circ\Gamma\circ I^{n-2}]_A]_A \\ &\subseteq (I\cup I^2\circ\Gamma\circ A\circ\Gamma\circ I^{n-2}]_A \\ &\subseteq (I\cup I^2\circ\Gamma\circ A\circ\Gamma\circ I^{n-2}]_2. \end{aligned}$$

Continuing this process in the similar fashion, we receive the following:

$$\begin{aligned} A\circ\Gamma\circ I^n &\subseteq (I\cup I^n\circ\Gamma\circ A\circ\Gamma\circ I^{n-n}]_A \\ &= (I\cup I^n\circ\Gamma\circ A]_A \\ &\subseteq (I]=I. \end{aligned}$$

So, $A\circ\Gamma\circ I^n\subseteq I$. Furthermore, let $i\in I$ and $a\in A$ be such that $a\leq i$. As $a\leq i$ and $i\in I$ where I is a relative $(n,0)$-Γ-hyperideal of H. This follows that $a\in I$. Therefore, I is a relative $(0,n)$-Γ-hyperideal of H. In a similar fashion, one can prove that every relative $(0,n)$-Γ-hyperideal of H is a relative $(n,0)$-Γ-hyperideal of H. Hence H is a relative n-duo ordered Γ-semihypergroup. □

Theorem 3.4. *Suppose that H is an ordered Γ-semihypergroup. Then H is a relative completely regular 2-duo ordered Γ-semihypergroup if and only if the following conditions hold for $S\subseteq H$:*

(1): $((I^2\cup I^2\circ\Gamma\circ S)^2]_S=I$ *for all relative $(0,2)$-Γ-hyperideal I of H.*
(2): $((P^2\cup S\circ\Gamma\circ P^2)^2]_S=P$ *for all relative $(2,0)$-Γ-hyperideal P of H.*

Proof. $\Rightarrow$ Suppose that H is a completely regular 2-duo ordered Γ-semihypergroup, we will show the condition (1). Let I be a relative $(0,2)$-Γ–hyperideal of H. Then, we have $I=(I^2]_S$ since $I\subseteq(I^2\circ\Gamma\circ I^2]_S\subseteq(I^2]_S\subseteq(I]_S=I$. For $I=(I^2]_S$, we find the following:

$$\begin{aligned} I &= (I^2]_S \\ &= ((I\cup A)^2]_S \\ &\subseteq (((I^2]_S\cup(I^2\circ\Gamma\circ S\circ\Gamma\circ I]_S)^2]_S \\ &\subseteq (I^2\cup I^2\circ\Gamma\circ S]_S^2]_S \\ &= ((I^2\cup I^2\circ\Gamma\circ S)^2]_S \\ &\subseteq (I^2]_S=I. \end{aligned}$$

Therefore, we have $((I^2\cup I^2\circ\Gamma\circ S)^2]_S=I$. The condition (2) can be proved in a similar fashion. $\Leftarrow$ Suppose that the conditions (1) and (2) hold true. Let I be a relative $(0,2)$-Γ-hyperideal of H. Then, we have the following:

$$\begin{aligned} I^2\circ\Gamma\circ S &= ((I^2\cup I^2\circ\Gamma\circ S)^2]_S\circ\Gamma\circ((I^2\cup I^2\circ\Gamma\circ S)^2]_S\circ\Gamma\circ(S]_S \\ &\subseteq ((I^2\cup I^2\circ\Gamma\circ S)^2]_S\circ\Gamma\circ(S]_S \\ &\subseteq ((I^2\cup I^2\circ\Gamma\circ S)\circ\Gamma\circ(I^2\cup I^2\circ\Gamma\circ S)\circ\Gamma\circ S]_S \\ &\subseteq ((I^2\cup I^2\circ\Gamma\circ S)\circ\Gamma\circ(I^2\circ\Gamma\circ S)]_S \\ &\subseteq ((I^2\cup I^2\circ\Gamma\circ S)\circ\Gamma\circ(I^2\cup I^2\circ\Gamma\circ S)]_S \\ &= ((I^2\cup I^2\circ\Gamma\circ S)^2]_S=I. \end{aligned}$$

Therefore, $I^2\circ\Gamma\circ S\subseteq I$. Obviously, if $i\in I$ and $s\in S$ such that $s\leq i$, then $s\in I$. Therefore, I is a (2, 0)-Γ-hyperideal of H. In a similar fashion, one can prove that P is a relative (0, 2)-Γ-hyperideal of H for all P is a relative (2, 0)-Γ-hyperideal of H. Thus, H is a 2-duo ordered Γ-semihypergroup.

Now, we have the following:

$$\begin{aligned}
i \in [i]_{2,0} &= (((([i]_{2,0})^2 \cup ([i]_{2,0})^2 \circ \Gamma \circ S)^2]_S \\
&= (((([i]_{2,0})^2 \cup ([i]_{2,0})^2 \circ \Gamma \circ S) \circ \Gamma \circ (([i]_{0,2})^2 \cup ([i]_{2,0})^2 \circ \Gamma \circ S)]_S \\
&= (((i \cup i^2 \cup i^2 \circ \Gamma \circ S]_S^2 \cup (i \cup i^2 \cup i^2 \circ \Gamma \circ S]_S^2 \circ \Gamma \circ S) \circ \Gamma \circ ((i \cup i^2 \cup S \circ \Gamma \circ i^2]^2 \cup ([i]_{2,0})^2 \circ \Gamma \circ S)]_S \\
&\subseteq (((i^2 \cup i^2 \circ \Gamma \circ S]_S \cup (i^2 \cup i^2 \circ \Gamma \circ S]_S \circ \Gamma \circ (S]) \circ \Gamma \circ ((i^2 \cup S \circ \Gamma \circ i^2] \cup [i]_{2,0}]_S \\
&\subseteq (((i^2 \cup i^2 \circ \Gamma \circ S]_S \cup (i^2 \circ \Gamma \circ S]_S) \circ \Gamma \circ ((i^2 \cup S \circ \Gamma \circ i^2]_S \cup [i]_{0,2})]_S \\
&= ((i^2 \cup i^2 \circ \Gamma \circ S]_S \circ \Gamma \circ (i \cup i^2 \cup S \circ \Gamma \circ S \circ i^2]_S]_S \\
&\subseteq (i^3 \cup i^4 \cup i^2 \circ \Gamma \circ S \circ \Gamma \circ i \cup i^2 \circ \Gamma \circ S \circ \Gamma \circ i^2]_S \\
&= (i^3] \cup (i^4] \cup (i^2 \circ \Gamma \circ S \circ \Gamma \circ i]_S \cup (i^2 \circ \Gamma \circ S \circ \Gamma \circ i]_S.
\end{aligned}$$

So, $i \in (i^3]$ or $i \in (i^4]$ or $i \in (i^2 \circ \Gamma \circ S \circ \Gamma \circ i]_S$ or $i \in (i^2 \circ \Gamma \circ S \circ \Gamma \circ i^2]_S$. If $i \in (i^3]_S$, then $i \leq i^3$. Thus, we find the following:

$$i \leq i^3 = i^2 \circ \Gamma \circ i \leq i^2 \circ \Gamma \circ i^3 = i^2 \circ \Gamma \circ i \circ \Gamma \circ i^2.$$

This follows that

$$i \in (i^2 \circ \Gamma \circ i \circ \Gamma \circ i^2]_S \subseteq (i^2 \circ \Gamma \circ S \circ \Gamma \circ i^2]_S.$$

In a similar fashion, if $i \in (i^4]_S$, then we get $i \in (i^2 \circ \Gamma \circ S \circ \Gamma \circ i^2]_S$. If $\imath \in (i^2 \circ \Gamma \circ S \circ \Gamma \circ i]_S$, then

$$\begin{aligned}
i \in (i^2 \circ \Gamma \circ S \circ \Gamma \circ (i^2 \circ \Gamma \circ S \circ \Gamma \circ i]_S]_S &\subseteq (i^2 \circ \Gamma \circ S \circ \Gamma \circ (i^2 \circ \Gamma \circ S \circ \Gamma \circ (i^2 \circ \Gamma \circ S \circ \Gamma \circ i]_S]_S]_S \\
&\subseteq (i^2 \circ \Gamma \circ S \circ \Gamma \circ i^2 \Gamma \circ S \circ \Gamma \circ i^2 \Gamma \circ S]_S.
\end{aligned}$$

As $(S \circ \Gamma \circ i^2]_S$ is a relative $(0,2)$-Γ-hyperideal of H and H is a 2-duo ordered Γ-semihypergroup, then $(S \circ \Gamma \circ i^2]_S$ is a relative $(2,0)$-Γ-hyperideal of H. This implies that $(S \circ \Gamma \circ i^2]^2 \circ \Gamma \circ S \subseteq (S \circ \Gamma \circ i^2]_S$. So, we have the following:

$$\begin{aligned}
i \in (i^2 \circ \Gamma \circ S \circ \Gamma \circ i^2 \circ \Gamma \circ S]_S) &\subseteq (i^2 \circ \Gamma \circ (S \circ \Gamma \circ i^2]_S \circ \Gamma \circ (S \circ \Gamma \circ i^2]_S \circ \Gamma \circ S]_S \\
&\subseteq (i^2 \circ \Gamma \circ (S \circ \Gamma \circ i^2]_S]_S \\
&\subseteq (i^2 \circ \Gamma \circ S \circ \Gamma \circ i^2]_S.
\end{aligned}$$

□

Hence in either case, H is a relative completely regular ordered Γ-semihypergroup.

4. Conclusion

Algebraic hyperstructures is being studied widely and is a very interesting field to work on for its future research directions. In this paper, we have shown that if H is a duo ordered Γ-semihypergroup, then the sets $L(S)$ and $R(S)$ coincide. Thereafter, we have proved that if H is a duo ordered Γ-semihypergroup, then H is an n-duo ordered Γ-semihypergroup where $n \geq 2$. We have also shed light on fetching necessary and sufficient conditions on relative completely regular 2-duo ordered Γ-semihypergroup for all relative $(0,2)$-Γ-hyperideal of H and for all relative $(2,0)$-Γ-hyperideal of H.

References

[1] A Basar, Shaista, B Satyanarayana and A Ahmad, On Relative Quasi-ideals and Relative Bi-ideals in AG-Rings, International Conference On "Exploring Excellence in Mathematical Sciences", 4-6 November, 2023, organized by Nalanda Open University, Nalanda in Collaboration with Bihar Mathematical Society(Submitted).

[2] A Basar, S Kumari and B Satyanarayana, On Some Properties Of Relative Hyperideals In Relative Regular Ordered Krasner Hyperrings, GIS Science Journal, 10(1)(2023), 306-328.

[3] A Basar, A Ahmad, B Satyanarayana, P K Sharma and M Sanjeel, On relative ordered Gamma-hyperideals and relative prime segments in chain ordered Gamma-semihypergroups, Journal of Xi'an University of Architecture & Technlogy, 14(8)(2022), 300-309.

[4] A Basar, A Ahmad, P. K. Sharma, A, S Bhavanari, On relative ordered Hoehnke Γ-hyperideal and Γ-hyperideals in chain ordered Γ-semihypegroups, GIS Science Journal, 9(7)(2022), 2793-2803.

[5] A Basar and B Davvaz, On relative quasi hyperideals and relative bi-hyperideals in ordered semihypergroups, Palestine Journal of Mathematics , 11(3)(2022) , 308–322.

[6] A Basar, B Satyanarayana, K. S. Prasad, P. K. Sharma and M. Y. Abbasi, A note on relative Gamma-ideals in abstract affine Gamma-nearrings, GIS SCIENCE JOURNAL, 8(10)(2021), 9-13.

[7] A Basar, On Some Relative Weakly Hyperideals and Relative Prime Bi-Hyperideals in Ordered Hypersemigroups and in Involution Ordered Hypersemigroups, ANNALS OF COMMUNICATIONS IN MATHEMATICS, 3(1)(2020),63-79.

[8] A. D. Wallace, Relative Ideals in Semigroups I, Colloq. Math., (1962), 55-61.

[9] A. D. Wallace, Relative Ideals in Semigroups II, Acta Mathematica Hungarica, 14(1-2)(1963), 137-148.

[10] A. Mahboob, N. M. Khan and B. Davvaz, Structural properties for (m,n) -quasi-hyperideals in ordered semihypergroups, Tbilisi Math. J. 11(4) (2018), 145-163.

[11] A. Mahboob, N. M. Khan and B. Davvaz, (m,n)-Hyperideals in ordered semihypergroups, Categ. Gen. Algebr. Struct. Appl. 12(1) (2020), 43-67.

[12] D. Heidari and B. Davvaz, Γ-Hypergroups and Γ-Semihypergroups Associated to Binary Relations. Iran. J. Sci. Technol. Trans. A Sci. 35 (2011), no. 2, 69–80.

[13] D. Heidari, S. Ostadhadi Dehkordi, and B. Davvaz, B. Γ-semihypergroups and their Properties. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 72 (2010), no. 1, 195–208.

[14] B. Davvaz, Semihypergroup Theory, Academic Press, 2016.

[15] D. Heidari and B. Davvaz, On ordered hyperstructures, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 73 (2) (2011), 85-96.

[16] D. Christoph, On quasi-ideals of semirings, International J. Math. and Math. Sci., 17(1) (1994), 47-58.

[17] F. Marty, Sur une generalization de la notion de groupe, 8eme ‘ Congres Math. Scandinaves, Stockholm, (1934), 45-49.

[18] K. Hila, B. Davvaz, and J. Dine, J. Study on the Structure of Γ-Semihypergroups, Commun. Algebra 40 (2012), no. 8, 2932–2948.

[19] H. J. Weinert, On quasi-ideals in rings, Acta Mathematica Hungarica, 43 (1984), 1–2.

[20] M. Kondo and N. Lekkoksung, On Intra-Regular Ordered Γ-Semihypergroups, Int. J. Math. Anal. (Ruse) 7 , 25-28(2013), 1379–1386.

[21] K. Iseki, Quasi-ideals in semirings without zero, Proc. Japan Acad., 34 (1958), 79–84.

[22] K. Iseki, Ideal in semirings, Proc. Japan Acad., 34 (1958), 29–31.

[23] K. Iseki, Ideal theory of semiring, Proc. Japan Acad., 32 (1956), 554–559.

[24] K. Hila, B. Davvaz and K. Naka, On Quasi-hyperideals in Semihypergroups, Communications in Algebra, 39 (2011), 4183-4194.

[25] L. K. Ardekani and B. Davvaz, Ordered semihypergroup constructions, Bol. Mat. 25(2) (2018), 77-99.

[26] N. M. Khan and M. F. Ali, Relative bi-ideals and relative quasi ideals in ordered semigroups, Hacet. J. Math. Stat., 20(10) (2019), 1–12.

[27] N. Kehayopulu, On completely regular ordered semigroups, Sci. Math., 1(1998), 27-32.

[28] N. Kehayopulu, S. Lajos and M. Singelis, On intra-regular ordered semigroups, PU. M. A. 4, 317-327, 1993.

[29] O. Steinfeld, On ideal-quotients and prime ideals, Acta. Math. Acad. Sci. Hung., 4(1953), 289-298.

[30] O. Steinfeld, Quasi ideals in Rings and Semigroups, vol. 10 of Disquisitiones Mathematicae Hungaricae, Akademiai Kiado, Budapest, Hungary, 1978.

[31] P. Corsini, Prolegomena of Hypergroup Theory, Aviani Editore, Italy, 1993.

[32] P. Corsini and V. Leoreanu, Applications of Hyperstructure Theory, Kluwer, Dordrecht, 2003.

[33] P. Corsini, Hypergraphs and Hypergroups. Algebra Universalis 35 (1996), no. 4, 548–555.

[34] P. Corsini, Hypergroups Associated with HX-groups. Analele S¸tiint¸. ale Univ. Ovidius Constant¸a Ser. Mat. 25 (2017), no. 2, 49–64.

[35] P. Luangchaisri and T. Changphas, On completely regular 2-duo semigroups, Quasigroups Relat. Syst. 29 (2021), 89-96.

[36] R.A. Good and D. R. Hughes, Associated groups for a semigroup. Bull. Amer. Math. Soc., (58)(1952), 624-625.

[37] S. Lajos, On (m,n)-ideals of semigroups. Abstracts of Second Hungar. Math. Congress I, (1960), 42-44.

[38] S. M. Anvariyeh, S. Mirvakili, B. Davvaz, On Γ-Hyperideals in Γ-Semihypergroups. Carpathian J. Math. 26(2010), no. 1, 11–23.

[39] S. J. Ansari, F. K. Pawar and K. Hila, On regular and intra-regular Γ-semihypergroups , CREAT. MATH. INFORM, 31(1), (2022) 17 - 31.

[40] V. N. Dixit and S. Dewan, A note on quasi and bi-ideals in ternary semigroups, Int. J. Math. Math. Sci., 18(1995), 501-508.

[41] W. Jantanan, P. Singavananda, A. Iampan, R. Chinram, On completely regular 2-duo ordered semihypergroups, Journal of Information & Optimization Sciences, 44 (2023), No. 5, pp. 895–908.