ON SOME DOUBLE GENERATING FUNCTIONS OF JACOBI POLYNOMIALS OF TWO VARIABLES

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Abstract: In this paper we obtained some double generating functions of Jacobi polynomials of two variables in terms of confluent hypergeometric functions of one and two variables, Horn functions and triple hypergeometric function respectively. .

Keywords: Jacobi polynomials of two variables, confluent hypergeometric functions, Horn functions, Generalized hypergeometric functions of three variables, Double generating functions.

Mathematics Subject Classification: 33C45

1. INTRODUCTION:

In 1991, S.F. Ragab [6] defined Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r L_{n-r}^{(\alpha)}(x)}{r!\,\Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)} \tag{1.1}$$

where $L_n^{(\alpha)}(x)$ is the well known Laguerre polynomials of one variable.

The definition (1.1) is equivalent to the following explicit representation of $L_n^{(\alpha,\beta)}(x,y)$ given by S.F. Ragab

$$L_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}x^s y^r}{(\alpha+1)_s(\beta+1)_r r!s!} \tag{1.2}$$

In the same year S.K.Chatterjea [2] obtained generating functions and many other interesting results for Ragab's Laguerre polynomial of two variables $L_n^{(\alpha,\beta)}(x,y)$. In 1997, M.A. Khan and A. K. Shukla [4] extended Laguerre polynomials of two variables to Laguerre polynomials of three variables and later to Laguerre polynomials of m-variables [5] and obtained many useful results.

In 1998, M.A. Khan and G. S. Abukhammash [3] defined Hermite polynomials of two variables $H_n(x,y)$ as follows:

$$H_n(x,y) = \sum_{r=0}^{[n/2]} \frac{n!(-y)^r H_{n-2r}(x)}{r!(n-2r)!} \tag{1.3}$$

where $H_n(x,y)$ is the well known Hermite polynomial of one variable.

In 2000, H.S.P. Shrivastava [9] defined Jacobi polynomials of two variables $P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y)$ as follows:

$$P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y) = \frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}$$
$$\times \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1+\alpha_1+\beta_1+n)_r(1+\alpha_2+\beta_2+n)_s}{r!s!(1+\alpha_1)_r(1+\alpha_2)_s} \left(\frac{1-x}{2}\right)^r \left(\frac{1-y}{2}\right)^s \tag{1.4}$$

They expressed the relation (1.4) also in terms of Appell's function of two variables as follows:

$$P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y) = \frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}$$
$$\times F_2\left[-n, 1+\alpha_1+\beta_1+n, 1+\alpha_2+\beta_2+n; 1+\alpha_1, 1+\alpha_2; \frac{1-x}{2}, \frac{1-y}{2}\right] \tag{1.5}$$

The definition (1.4) can also be represented as follows:

$$P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y) = \frac{\Gamma(1+\alpha_1+n)\Gamma(1+\alpha_2+n)}{n!}$$
$$\times \sum_{r=0}^{n} \frac{(-1)^r(1+\alpha_1+\beta_1+n)_r\left(\frac{1-x}{2}\right)^r}{r!\Gamma(1+\alpha_1+r)\Gamma(1+\alpha_2+n-r)} P_{n-r}^{(\alpha_2,\,\beta_2+r)}(y) \tag{1.6}$$

where $P_n^{(\alpha,\,\beta)}(y)$ is the well known Jacobi polynomial of one variable.

On the basis of above study we obtained some generating functions of Jacobi polynomials of two variables in our earlier paper [1]. In extension to this in this paper we obtained some double generating functions of Jacobi polynomials of two variables in terms of confluent hypergeometric functions of one and two variables, Horn functions and triple hypergeometric function respectively.

In our study we require the following definitions:

The confluent hypergeometric function of one variable is defined by [7]

$$ {}_1F_1(a;\ c;\ z) \ = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\, \frac{z^n}{n!}. \tag{1.7}$$

The confluent hypergeometric functions of two variables are defined by [8]

$$\phi_1\left[\alpha,\beta;\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}\ , \qquad |x|<1,\quad |y|<\infty\ ; \tag{1.8}$$

$$\phi_2\left[\beta,\beta';\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}\ , \qquad |x|<\infty,\quad |y|<\infty\ ; \tag{1.9}$$

$$\phi_3\left[\beta;\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}\ , \qquad |x|<\infty,\quad |y|<\infty\ ; \tag{1.10}$$

$$\Psi_1\left[\alpha,\beta;\gamma,\gamma';x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!}\frac{y^n}{n!}\ , \qquad |x|<1,\quad |y|<\infty\ ; \tag{1.11}$$

$$\Psi_2\left[\alpha;\gamma,\gamma';x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!}\frac{y^n}{n!}\ , \qquad |x|<\infty,\quad |y|<\infty\ ; \tag{1.12}$$

$$\Xi_1\left[\alpha,\alpha',\beta;\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}\ , \quad |x|<1,\quad |y|<\infty\ . \tag{1.13}$$

The Horn functions of two variables are defined by [8]

$$G_1\left[\alpha,\beta,\beta';x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (\alpha)_{m+n}(\beta)_{n-m}(\beta')_{m-n} \frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.14}$$

$$|x|<r, \qquad |y|<s, \qquad r+s=1;$$

$$G_2\left[\alpha,\alpha',\beta,\beta';x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.15}$$

$$|x|<1, \quad |y|<1;$$

$$G_3\left[\alpha,\alpha';x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2n-m}(\alpha')_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.16}$$

$$|x|<r, \quad |y|<s, \quad 27r^2s^2+18rs\pm 4(r-s)-1=0;$$

$$H_1\left[\alpha,\beta,\gamma;\delta;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.17}$$

$$|x|<r, \quad |y|<s, \quad 4rs=(s-1)^2;$$

$$H_2\left[\alpha,\beta,\gamma,\delta;\epsilon;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\epsilon)_m}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.18}$$

$$|x|<r, \quad |y|<s, \quad (r+1)s=1;$$

$$H_3\left[\alpha,\beta;\gamma;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.19}$$

$$|x|<r, \quad |y|<s, \quad r+(s-\tfrac{1}{2})^2=\tfrac{1}{4};$$

$$H_4\left[\alpha,\beta;\gamma,\delta;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.20}$$

$$|x|<r, \quad |y|<s, \quad 4r=(s-1)^2;$$

$$H_5\left[\alpha,\beta;\gamma;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_{n-m}}{(\gamma)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.21}$$

$$|x|<r, \quad |y|<s, \quad 16r^2-36rs\pm(8r-s+27rs^2)+1=0;$$

$$H_6\left[\alpha,\beta,\gamma;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.22}$$

$$|x|<r, \quad |y|<s, \quad rs^2+s-1=0;$$

$$H_7\left[\alpha,\beta,\gamma;\delta;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m-n}(\beta)_n(\gamma)_n}{(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.23}$$
$$|x|<r,\qquad |y|<s,\qquad 4r=(s^{-1}-1)^2.$$

The generalized hypergeometric function of three variables $F^3[x,y,z]$ are defined by [8]

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');\ (b''):(c);\ (c');(c'');\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{array}x,y,z\right]$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{m+p}((c))_m((c'))_n((c''))_p}{((e))_{m+n+p}((g))_{m+n}((g'))_{n+p}((g''))_{m+p}((h))_m((h'))_n((h''))_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{1.24}$$

2. DOUBLE GENERATING RELATIONS INVOLVING CONFLUENT HYPERGEOMETRIC FUNCTIONS:

Certain modifications of the sequence $\{P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y)\}_{n\in N}$ admit the following double generating relations in terms of confluent hypergeometric functions:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n)}(x,y)u^mt^n=e^t(1-u)^{-1-\alpha_1-\beta_1}$$
$$\times{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{\frac{1}{2}(x-1)t}{(1-u)}\right]{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right], \tag{2.1}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n)}(x,y)u^mt^n=e^t(1-u)^{-1-\alpha_2-\beta_2}$$
$$\times{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{\frac{1}{2}(y-1)t}{(1-u)}\right], \tag{2.2}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(x,y)u^mt^n$$
$$=e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\phi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.3}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-n)}(x,y)u^mt^n$$
$$=e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\phi_1\left[1+\alpha_2+\beta_2,\lambda;1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.4}$$

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$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n)}(x,y)u^m t^n$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\phi_2\left[\lambda,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \quad (2.5)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n)}(x,y)u^m t^n$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\phi_2\left[\lambda,1+\alpha_2+\beta_2;1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \quad (2.6)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n)}(x,y)u^m t^n$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right], \quad (2.7)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n)}(x,y)u^m t^n$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,u\right], \quad (2.8)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n)}(x,y)u^m t^n$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\Psi_1\left[1+\alpha_1+\beta_1,\lambda;\mu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \quad (2.9)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_2+\beta_2)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n)}(x,y)u^m t^n$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\Psi_1\left[1+\alpha_2+\beta_2,\lambda;\mu,1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \quad (2.10)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n)}(x,y)u^m t^n$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\Psi_2\left[1+\alpha_1+\beta_1;\mu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \quad (2.11)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n)}(x,y)u^m t^n$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\Psi_2\left[1+\alpha_2+\beta_2;\mu,1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \quad (2.12)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n)}(x,y)u^m t^n$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \quad (2.13)$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n)}(x,y)u^m t^n$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\Xi_1\left[\lambda,1+\alpha_2+\beta_2,\mu;1+\alpha_2;u,\frac{1}{2}(y-1)t\right]. \quad (2.14)$$

PROOF OF (2.1): Starting with left-side of (2.1) and using the definition (1.4) of Jacobi polynomials of two variables, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s}{m!n!r!s!(1+\alpha_1)_r(1+\alpha_2)_s}\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s u^m t^n$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s}{m!n!r!s!(1+\alpha_1)_r(1+\alpha_2)_s}\left\{\frac{1}{2}(x-1)t\right\}^r\left\{\frac{1}{2}(y-1)t\right\}^s u^m t^n$$

$$= e^t\,(1-u)^{-1-\alpha_1-\beta_1}{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{\frac{1}{2}(x-1)t}{(1-u)}\right]{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$

where ${}_1F_1$ is confluent hypergeometric function defined by eq. (1.7), which proves (2.1). The proof of result (2.2) is similar to that of (2.1).

PROOF OF (2.3): Starting with left-side of (2.3) and using the definition (1.4) of Jacobi polynomials of two variables, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(\lambda)_m(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s}{m!n!r!s!(1+\alpha_1)_{m+r}(1+\alpha_2)_s}\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s u^m t^n$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\lambda)_m(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s}{m!n!r!s!(1+\alpha_1)_{m+r}(1+\alpha_2)_s} \left\{\frac{1}{2}(x-1)t\right\}^r \left\{\frac{1}{2}(y-1)t\right\}^s u^m t^n$$

$$= e^t {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right] \phi_1\left[1+\alpha_1+\beta_1, \lambda; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

where ϕ_1 is (Humbert's) confluent hypergeometric function of two variables defined by eq. (1.8), which proves (2.3).
The proof of results (2.4) to (2.14) are similar to that of (2.3).

3. DOUBLE GENERATING RELATIONS INVOLVING HORN FUNCTIONS:

Certain modifications of the sequence $\{P_n^{(\alpha_1,\, \beta_1;\, \alpha_2,\, \beta_2)}(x,y)\}_{n\in N}$ admit the following double generating relations in terms of Horn functions of two variables:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(-\alpha_1)_m}{m!(-\alpha_2-\beta_2)_m(1+\alpha_2)_{n-m}} P_n^{(\alpha_1-m-n,\, \beta_1+m;\, \alpha_2-m,\, \beta_2-n)}(x,y)u^m t^n = (1+t)^{\alpha_1}$$

$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1} G_1\left[-\alpha_1, 1+\alpha_2+\beta_2, -\alpha_2; \frac{u}{1+t}, \frac{\frac{1}{2}(y-1)t}{1+t}\right], \tag{3.1}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(-\alpha_2)_m}{m!(-\alpha_1-\beta_1)_m(1+\alpha_1)_{n-m}} P_n^{(\alpha_1-m,\, \beta_1-n;\, \alpha_2-m-n,\, \beta_2+m)}(x,y)u^m t^n = (1+t)^{\alpha_2}$$

$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2} G_1\left[-\alpha_2, 1+\alpha_1+\beta_1, -\alpha_1; \frac{u}{1+t}, \frac{\frac{1}{2}(x-1)t}{1+t}\right], \tag{3.2}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m}{m!(-\alpha_2-\beta_2)_m(1+\alpha_2)_{n-m}} P_n^{(\alpha_1-n,\, \beta_1;\, \alpha_2-m,\, \beta_2-n)}(x,y)u^m t^n = (1+t)^{\alpha_1}$$

$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1} G_2\left[\lambda, -\alpha_1, 1+\alpha_2+\beta_2, -\alpha_2;\ u, \frac{\frac{1}{2}(y-1)t}{1+t}\right], \tag{3.3}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m}{m!(-\alpha_1-\beta_1)_m(1+\alpha_1)_{n-m}} P_n^{(\alpha_1-m,\, \beta_1-n;\, \alpha_2-n,\, \beta_2)}(x,y)u^m t^n = (1+t)^{\alpha_2}$$

$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2} G_2\left[\lambda, -\alpha_2, 1+\alpha_1+\beta_1, -\alpha_1;\ u, \frac{\frac{1}{2}(x-1)t}{1+t}\right], \tag{3.4}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_{n-m}}{m!(1+\alpha_2)_{n-2m}(1+\alpha_1)_n}P_n^{(\alpha_1,\,-n;\,\alpha_2-2m,\,\beta_2+m)}(x,y)u^m t^n$$

$$=\left\{1-\frac{1}{2}(x+1)t\right\}^{-1-\alpha_2-\beta_2}\ G_3\left[1+\alpha_2+\beta_2,-\alpha_2;\ u\left(1-\frac{1}{2}(x+1)t\right),\frac{\frac{1}{2}(1-y)t}{1-\frac{1}{2}(x+1)t}\right],\quad (3.5)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_{n-m}}{m!(1+\alpha_1)_{n-2m}(1+\alpha_2)_n}P_n^{(\alpha_1-2m,\,\beta_1+m;\,\alpha_2,\,-n)}(x,y)u^m t^n$$

$$=\left\{1-\frac{1}{2}(y+1)t\right\}^{-1-\alpha_1-\beta_1}\ G_3\left[1+\alpha_1+\beta_1,-\alpha_1;\ u\left(1-\frac{1}{2}(y+1)t\right),\frac{\frac{1}{2}(1-x)t}{1-\frac{1}{2}(y+1)t}\right],\quad (3.6)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_m(1+\alpha_2)_n}P_n^{(\alpha_1+m-n,\,\beta_1-m;\,\alpha_2,\,\beta_2+m-n)}(x,y)u^m t^n=(1+t)^{\alpha_1}$$

$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1}\ H_1\left[-\alpha_1,1+\alpha_2+\beta_2,\lambda;1+\alpha_2;\frac{\frac{1}{2}(1-y)t}{1+t},-u(1+t)\right],\quad (3.7)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_m}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+m-n,\,\beta_2-m)}(x,y)u^m t^n=(1+t)^{\alpha_2}$$

$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2}\ H_1\left[-\alpha_2,1+\alpha_1+\beta_1,\lambda;1+\alpha_1;\frac{\frac{1}{2}(1-x)t}{1+t},-u(1+t)\right],\quad (3.8)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_m(1+\alpha_2)_n}P_n^{(\alpha_1+m-n,\,\beta_1-m;\,\alpha_2,\,\beta_2-n)}(x,y)u^m t^n=(1+t)^{\alpha_1}$$

$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1}\ H_2\left[-\alpha_1,1+\alpha_2+\beta_2,\lambda,\mu;1+\alpha_2;\frac{\frac{1}{2}(1-y)t}{1+t},-u(1+t)\right],\quad (3.9)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_m}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m-n,\,\beta_2-m)}(x,y)u^m t^n=(1+t)^{\alpha_2}$$

$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2}\ H_2\left[-\alpha_2,1+\alpha_1+\beta_1,\lambda,\mu;1+\alpha_1;\frac{\frac{1}{2}(1-x)t}{1+t},-u(1+t)\right],\quad (3.10)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_{2m}}{m!(1+\alpha_2)_{m+n}}P_n^{(\alpha_1-n,\,\beta_1;\,\alpha_2+m,\,\beta_2+m-n)}(x,y)u^m t^n=(1+t)^{\alpha_1}$$
$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1}H_3\left[1+\alpha_2+\beta_2,-\alpha_1;1+\alpha_2;\ u,\frac{\frac{1}{2}(1-y)t}{1+t}\right], \tag{3.11}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_{2m}}{m!(1+\alpha_1)_{m+n}}P_n^{(\alpha_1+m,\,\beta_1+m-n;\,\alpha_2-n,\,\beta_2)}(x,y)u^m t^n=(1+t)^{\alpha_2}$$
$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2}H_3\left[1+\alpha_1+\beta_1,-\alpha_2;1+\alpha_1;\ u,\frac{\frac{1}{2}(1-x)t}{1+t}\right], \tag{3.12}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_{2m}}{m!(\lambda)_m(1+\alpha_2)_n}P_n^{(\alpha_1-n,\,\beta_1;\,\alpha_2,\,\beta_2+2m-n)}(x,y)u^m t^n=(1+t)^{\alpha_1}$$
$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1}H_4\left[1+\alpha_2+\beta_2,-\alpha_1;\lambda,1+\alpha_2;\ u,\frac{\frac{1}{2}(1-y)t}{1+t}\right], \tag{3.13}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_{2m}}{m!(\lambda)_m(1+\alpha_1)_n}P_n^{(\alpha_1,\,\beta_1+2m-n;\,\alpha_2-n,\,\beta_2)}(x,y)u^m t^n=(1+t)^{\alpha_2}$$
$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2}H_4\left[1+\alpha_1+\beta_1,-\alpha_2;\lambda,1+\alpha_1;\ u,\frac{\frac{1}{2}(1-x)t}{1+t}\right], \tag{3.14}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_{2m}}{m!(1+\alpha_1)_m(1+\alpha_2)_n}P_n^{(\alpha_1+m-n,\,\beta_1-m;\,\alpha_2,\,\beta_2+2m-n)}(x,y)u^m t^n=(1+t)^{\alpha_1}$$
$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1}H_5\left[1+\alpha_2+\beta_2,-\alpha_1;1+\alpha_2;\ -u(1+t),\frac{\frac{1}{2}(1-y)t}{1+t}\right], \tag{3.15}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_{2m}}{m!(1+\alpha_1)_n(1+\alpha_2)_m}P_n^{(\alpha_1,\,\beta_1+2m-n;\,\alpha_2+m-n,\,\beta_2-m)}(x,y)u^m t^n=(1+t)^{\alpha_2}$$
$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2}H_5\left[1+\alpha_1+\beta_1,-\alpha_2;1+\alpha_1;\ -u(1+t),\frac{\frac{1}{2}(1-x)t}{1+t}\right], \tag{3.16}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(-\alpha_2-\beta_2)_m(1+\alpha_2)_{n-2m}}P_n^{(\alpha_1-n,\,\beta_1;\,\alpha_2-2m,\,\beta_2+m-n)}(x,y)u^m t^n=(1+t)^{\alpha_1}$$
$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1}H_6\left[-\alpha_2,1+\alpha_2+\beta_2,-\alpha_1;-u,\frac{\frac{1}{2}(y-1)t}{1+t}\right], \quad (3.17)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(-\alpha_1-\beta_1)_m(1+\alpha_1)_{n-2m}}P_n^{(\alpha_1-2m,\,\beta_1+m-n;\,\alpha_2-n,\,\beta_2)}(x,y)u^m t^n=(1+t)^{\alpha_2}$$
$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2}H_6\left[-\alpha_1,1+\alpha_1+\beta_1,-\alpha_2;-u,\frac{\frac{1}{2}(x-1)t}{1+t}\right], \quad (3.18)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(\lambda)_m(1+\alpha_2)_{n-2m}}P_n^{(\alpha_1-n,\,\beta_1;\,\alpha_2-2m,\,\beta_2+2m-n)}(x,y)u^m t^n=(1+t)^{\alpha_1}$$
$$\times\left\{1-\frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1}H_7\left[-\alpha_2,1+\alpha_2+\beta_2,-\alpha_1;\lambda;u,\frac{\frac{1}{2}(y-1)t}{1+t}\right], \quad (3.19)$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(\lambda)_m(1+\alpha_1)_{n-2m}}P_n^{(\alpha_1-2m,,\,\beta_1+2m-n;\,\alpha_2-n\,\beta_2)}(x,y)u^m t^n=(1+t)^{\alpha_2}$$
$$\times\left\{1-\frac{1}{2}(y-1)t\right\}^{-1-\alpha_2-\beta_2}H_7\left[-\alpha_1,1+\alpha_1+\beta_1,-\alpha_2;\lambda;u,\frac{\frac{1}{2}(x-1)t}{1+t}\right]. \quad (3.20)$$

PROOF OF (3.1): Consider the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(-\alpha_1)_m}{m!(-\alpha_2-\beta_2)_m(1+\alpha_2)_{n-m}}P_n^{(\alpha_1-m-n,\,\beta_1+m;\,\alpha_2-m,\,\beta_2-n)}(x,y)u^m t^n$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(-\alpha_1)_m}{m!(-\alpha_2-\beta_2)_m(1+\alpha_2-m)_n(1+\alpha_2)_{-m}}\times\frac{(1+\alpha_1-m-n)_n(1+\alpha_2-m)_n}{(n!)^2}$$
$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2-m)_s}{r!s!(1+\alpha_1-m-n)_r(1+\alpha_2-m)_s}\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s t^n u^m$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_{s-m}}{m!r!s!(n-r-s)!(1+\alpha_1)_{r-m-n}(1+\alpha_2)_{s-m}}\left(\frac{x-1}{2}\right)^r\left(\frac{y-1}{2}\right)^s t^n u^m$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_{s-m}(-\alpha_1)_{m+n+s}(-\alpha_2)_{m-s}}{m!n!r!s!}$$
$$\times\left\{\frac{1}{2}(x-1)t\right\}^r\left\{\frac{1}{2}(y-1)t\right\}^s(-t)^n u^m$$

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$$= (1+t)^{\alpha_1} \left\{1 - \frac{1}{2}(x-1)t\right\}^{-1-\alpha_1-\beta_1} G_1\left[-\alpha_1, 1+\alpha_2+\beta_2, -\alpha_2; \frac{u}{1+t}, \frac{\frac{1}{2}(y-1)t}{1+t}\right]$$

where G_1 is defined by eq. (1.14), which proves (3.1).
The proofs of results (3.2) to (3.20) are similar to that of (3.1).

4. DOUBLE GENERATING RELATIONS INVOLVING TRIPLE HYPERGEOMETRIC FUNCTION:

Certain modifications of the sequence $\{P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y)\}_{n\in N}$ admit the following double generating relations in terms of triple hypergeometric function:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n} P_n^{(\alpha_1,\beta_1+m-n;\alpha_2,\beta_2+m-n)}(x,y)u^m t^n$$

$$= e^t\ F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : -; \quad - ; \quad - ; \\ - :: \quad - \quad ; -; \quad - \quad : -; 1+\alpha_1; 1+\alpha_2; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \quad (4.1)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1+m)_n(1+\alpha_2+m)_n} P_n^{(\alpha_1+m,\beta_1-n;\alpha_2+m,\beta_2-n)}(x,y)u^m t^n$$

$$= e^t\ F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : 1+\alpha_1, 1+\alpha_2; -; -; \\ - :: \quad 1+\alpha_1 \quad ; -; \quad 1+\alpha_2 \quad : \quad - \quad ; -; -; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \quad (4.2)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}} P_n^{(\alpha_1+m,\beta_1-n;\alpha_2+m,\beta_2-n)}(x,y)u^m t^n$$

$$= e^t\ F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : -; -; -; \\ - :: \quad 1+\alpha_1 \quad ; -; \quad 1+\alpha_2 \quad : -; -; -; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \quad (4.3)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!}{m!(1+\alpha_1+m)_n(1+\alpha_2+m)_n} P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-n)}(x,y)u^m t^n$$

$$= e^t\ F^{(3)}\left[\begin{matrix} - :: \quad - \quad ; -; 1+\alpha_2+\beta_2 : 1+\alpha_1, 1+\alpha_2; 1+\alpha_1+\beta_1; -; \\ - :: 1+\alpha_1; -; \quad 1+\alpha_2 \quad : \quad 1+\alpha_2+\beta_2\ ; \quad - \quad ; -; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \quad (4.4)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}}P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: \quad - \ ;-;1+\alpha_2+\beta_2 : -;1+\alpha_1+\beta_1;-; \\ - :: 1+\alpha_1;-; \quad 1+\alpha_2 \ : -; \quad - \ ;-; \end{array} u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \tag{4.5}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(1+\alpha_1+m)_n(1+\alpha_2+m)_n}P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-m-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: \quad - \ ;-; \quad - \ :1+\alpha_1,1+\alpha_2;1+\alpha_1+\beta_1;1+\alpha_2+\beta_2; \\ - :: 1+\alpha_1;-;1+\alpha_2 : \quad - \ ; \quad - \ ; \quad - \ ; \end{array} u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \tag{4.6}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}}P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-m-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: \quad - \ ;-; \quad - \ :\lambda;1+\alpha_1+\beta_1;1+\alpha_2+\beta_2; \\ - :: 1+\alpha_1;-;1+\alpha_2 : -; \quad - \ ; \quad - \ ; \end{array} u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \tag{4.7}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}}P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: \quad - \ ;-;1+\alpha_2+\beta_2 : 1+\alpha_1,1+\alpha_2;1+\alpha_1+\beta_1;-; \\ - :: 1+\alpha_1;-; \quad 1+\alpha_2 \ : \ 1+\alpha_2+\beta_2 \ ; \quad - \ ; -; \end{array} u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \tag{4.8}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}}P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-m-n)}(x,y)u^m t^n$$
$$= e^t F^{(3)}\left[\begin{array}{l} - :: \quad - \ ;-; \quad - \ : -;1+\alpha_1+\beta_1;1+\alpha_2+\beta_2; \\ - :: 1+\alpha_1;-;1+\alpha_2 : -; \quad - \ ; \quad - \ ; \end{array} u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \tag{4.9}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\beta_1+m-n;\alpha_2,\beta_2+m-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: 1+\alpha_1+\beta_1;-;1+\alpha_2+\beta_2 : \quad \lambda \ ; \quad - \ ; \quad - \ ; \\ - :: \quad - \ ;-; \quad - \ : 1+\alpha_1+\beta_1,1+\alpha_2+\beta_2;1+\alpha_1;1+\alpha_2; \end{array}\right.$$
$$\left. u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \tag{4.10}$$

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$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\beta_1+m-n;\alpha_2,\beta_2+m-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : \lambda; \quad - ; \quad - ; \\ - :: \quad - \quad ; -; \quad - \quad : \mu; 1+\alpha_1; 1+\alpha_2; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \quad (4.11)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\beta_1+m-n;\alpha_2,\beta_2+m-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : \quad - \quad ; \quad - ; \quad - ; \\ - :: \quad - \quad ; -; \quad - \quad : 1+\alpha_1+\beta_1, 1+\alpha_2+\beta_2; 1+\alpha_1; 1+\alpha_2; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \quad (4.12)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\beta_1+m-n;\alpha_2,\beta_2+m-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : -; \quad - ; \quad - ; \\ - :: \quad - \quad ; -; \quad - \quad : -; 1+\alpha_1; 1+\alpha_2; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \quad (4.13)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}}P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: \quad - \quad ; -; 1+\alpha_2+\beta_2 : \quad \lambda, \mu \quad ; 1+\alpha_1+\beta_1; -; \\ - :: 1+\alpha_1; -; \quad 1+\alpha_2 \quad : 1+\alpha_2+\beta_2; \quad - \quad ; -; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right] \quad (4.14)$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}}P_n^{(\alpha_1+m,\beta_1-m-n;\alpha_2+m,\beta_2-m-n)}(x,y)u^m t^n$$
$$= e^t\ F^{(3)}\left[\begin{array}{l} - :: \quad - \quad ; -; \quad - \quad : \lambda, \mu; 1+\alpha_1+\beta_1; 1+\alpha_2+\beta_2; \\ - :: 1+\alpha_1; -; \ 1+\alpha_2 \quad : \quad -; \quad - \quad ; \quad - \quad ; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right]. \quad (4.15)$$

PROOF OF (4.1): Starting with left-side of (4.1) and using the definition (1.4) of Jacobi polynomials of two variables, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{m+s}}{m!n!r!s!(1+\alpha_1)_r(1+\alpha_2)_s}\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s u^m t^n$$

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$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{m+s}}{m!n!r!s!(1+\alpha_1)_r(1+\alpha_2)_s} \left\{\frac{1}{2}(x-1)t\right\}^r \left\{\frac{1}{2}(y-1)t\right\}^s u^m t^n$$

$$= e^t F^{(3)} \left[\begin{array}{l} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : -; \quad - ; \quad - ; \\ - :: \quad - \quad ; -; \quad - \quad : -; 1+\alpha_1; 1+\alpha_2; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t \right]$$

where $F^3[x,y,z]$ is defined by eq. (1.24), which proves (4.1).
The proof of results (4.2) to (4.15) are similar to that of (4.1).

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