Line annihilator graph and generalized outer planarity index of a graph

S. Karthik [1] and K. Selvakumar[2]

[1]Department of Mathematics, Sardar Raja Arts and Science College, Vadakkangulam.
[2]Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli.

Abstract

In this paper, we study the annihilator graph is a line graph and the annihilator graph is the complement of a line graph. We give a full characterization of these graphs with respect to their planar and outerplanar indices. Also, we determine of these graphs respect to their generalized outerplanar index.

Key words and phrases: annihilator graph; line graph; planar graph; outer planar.
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1 Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has gained considerable attention. The research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other.

In the literature, one can find a number of different types of graphs attached to rings or other algebraic structures. The concept of the zero-divisor graph of a commutative ring, denoted by $\Gamma(R)$, was introduced by Beck [7], where he was mainly interested in coloring. This investigation of coloring of the zero-divisor graph of a commutative ring was then continued by Anderson and Naseer. The above definition later appeared in [5], which contained several fundamental results concerning the graph $\Gamma(R)$. The zero-divisor graphs of commutative rings has been studied by several authors. A similar work on the zero-divisors done by Badawi [6]. He defined the annihilator graph $AG(R)$ of a commutative ring R, which is a graph whose vertices are $Z(R)^*$ and two vertices x and y are joined by an edge if and only if $ann(xy) \neq ann(x) \cup ann(y)$. Several authors studied about various properties of these graphs including diameter, girth, domination and genus. In this paper, we study the annihilator graph is a line graph and the annihilator graph is the complement of a line graph. We give a full characterization of these graphs with respect to their planar and outerplanar indices. Also, we determine of these graphs respect to their generalized outerplanar index.

Let G be a graph. We use the notation $V(G)$ and $E(G)$ for vertex set and edge set of G, respectively. Also, P_n and C_n are used to denote a path and a cycle on n vertices, respectively. A graph is complete if every vertex is adjacent to every other vertex. The complete graph with n vertices is denoted by K_n. A bipartite graph is a graph which its vertex set is a union of two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2. A bipartite graph is complete bipartite if the vertex in each part is connected to every vertex in the other part. We denote the complete bipartite graph with parts of size m and n by $K_{m,n}$. we recall G is a planar graph if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The graph G is an outerplanar graph if it can be drawn on the plane without crossings in such a way that all of the vertices belong to the external face of the drawing. Also, G is a generalized outerplanar graph if it can be drawn on the plane in such a way that at least one end-vertex of each edge lies on the external face.

The line graph of a graph G, denoted by $L(G)$, is defined as a graph in which each vertex represents an edge of G and two vertices are adjacent if and only if their corresponding edges share a common endpoint in G. In recent years, the investigation of iterated line graphs has recorded a large progress. The k^{th} iterated line graph of G is denoted by $L^k(G)$ and these graphs are defined inductively as follows: $L^0(G) = G, L^1(G) = L(G)$ is the line graph of G and $L^k(G) = L(L^{k-1}(G))$. The planarity index of graph G was defined as the smallest k such that $L^k(G)$ is non-planar. We denote the planarity index of G by $\xi(G)$. If $L^k(G)$ is planar for all $k \geq 0$, we define $\xi(G) = \infty$. It was shown in [16] that if G is non-planar then $L(G)$ is also non-planar.

Through out this paper, we assume that R finite commutative ring with identity, $Z(R)^*$ is set of all non- zero divisor of R, $U(R)$ its group of units, $\mathbb{F}_q$ denote the field with q elements. Furthermore, for the convenience of the reader, we state without proof a few known results in the form of theorems, which will be used in the proofs of the main theorems.

Theorem 1.1. [6, Theorem 3.10] *Let R be an non-reduced commutative ring with $|Nil(R)^*| \geq 2$ and let $AG_N(R)$ be the (induced) subgraph of $AG(R)$ with vertices $Nil(R)^*$. Then $AG_N(R)$ is complete.*

Theorem 1.2. [9, Theorem 1.1] *The line graph of a graph G is planar if and only if G is planar, $\Delta(G) \leq 4$, and every vertex of degree 4 in G is a cut-vertex.*

Theorem 1.3. [20, Theorem 14] Let $(R, \mathfrak{m})$ be a finite commutative local ring with identity. Then $AG(R)$ is planar if and only if R is isomorphic to one of the following 13 rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x,2\rangle^2}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$, $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$.

Theorem 1.4. [20, Theorem 15] Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be a finite commutative non- local ring, where each R_i is a local ring and $n \geq 2$. Then $AG(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{F}$ is finite field.

2 When $AG(R)$ and $\overline{AG(R)}$ are line graph

In this section, we classify all finite commutative rings, whose annihilator graph and its complements are line graph. In order to do this, we will use one of the characterizations of line graphs which was proved in [8].

Theorem 2.1. [8] *Let G be a graph. Then G is the line graph of some graph if and only if none of the nine graphs in Fig. 1 is an induced subgraph of G.*

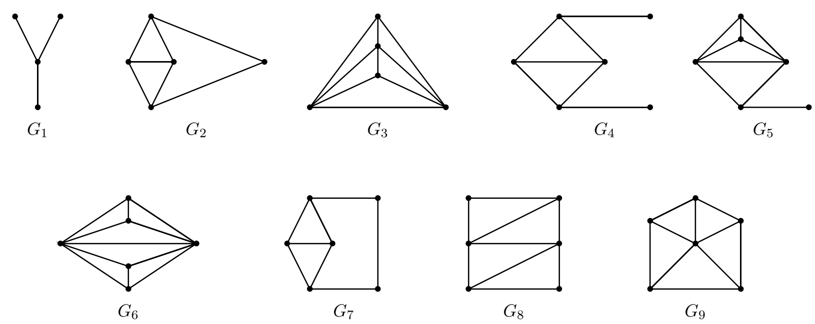

Fig.1. Forbidden induced subgraphs of line graphs

Theorem 2.2. *Let R be a finite commutative ring which is not a field. Then $AG(R)$ is a line graph if and only if R is isomorphic to one of the rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Suppose $AG(R)$ is a line graph. Since R is a finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$ where R_i is a local ring for all $i = 1, 2, \ldots, n$.

Suppose $n \geq 4$. Then it is easy to see that the subgraph induced by the set $\{(1,0,0,0,\cdots,0),(0,0,1,1,\cdots,0),(0,0,0,1,\cdots,0),(0,1,1,1,\cdots,0)\}$ in $AG(R)$ is isomorphic to $K_{1,3}$ and hence $K_{1,3}$ is a subgraph of $AG(R)$. By Theorem 2.1, $AG(R)$ is not a line graph, a contradiction. Thus $n \leq 3$.

Suppose $n = 3$ and $R \cong R_1 \times R_2 \times R_3$. Suppose that one of the rings R_i has at least 3 elements. Without loss of generality, we assume that $|R_3| \geq 3$. Let $a \in R_3$ be an arbitrary element such that $a \notin \{0, 1\}$. It is easy to see that the induced subgraph by the set $\{(1,0,0),(0,1,1),(0,1,a),(0,0,1)\}$ is isomorphic to $K_{1,3}$. Hence the graph $\mathbb{AG}(R_1 \times R_2 \times R_3)$ is not a line graph if one of the ring R_i has at least 3 elements. Hence R_i are fields with $|R_i| = 2$ for all $1 \leq i \leq 3$. Thus can easily see that the graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is the line graph of the graph $K_{2,3}$.

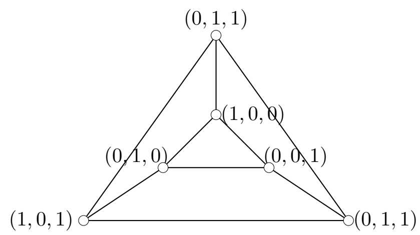

Fig. 2. $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Suppose $n = 2$ and $R \cong R_1 \times R_2$. Suppose that at least one of R_i is not a field. Without loss of generality, we assume that R_1 is not a field. Then $|R_1| \geq 4$ and so we can find a copy of $K_{1,3}$ as an induced subgraph in the graph $AG(R_1 \times R_2)$ and so $AG(R_1 \times R_2)$is not a line graph, a contradiction. So, R_i are fields and hence R_1 and R_2 has at most 3 elements. Thus R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. It is easy to see that $AG(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong P_2$ and it is the line graph of the graph P_3, $AG(\mathbb{Z}_2 \times \mathbb{Z}_3) \cong P_3$ and it is the line graph of the graph P_4 and $AG(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong C_4$ and it is the line graph of the graph C_4.

Finally, if $n = 1$, then R is local ring and by Theorem 1.1, $AG(R)$ is complete. Hence $AG(R)$ is a line graph of $K_{1,m}$, where $m = |Z(R)^*|$. □

Theorem 2.3. [8] *A graph G is the complement of a line graph if and only if none of the nine graphs $\overline{G_i}$ of Fig. 3. is an induced subgraph of G.*

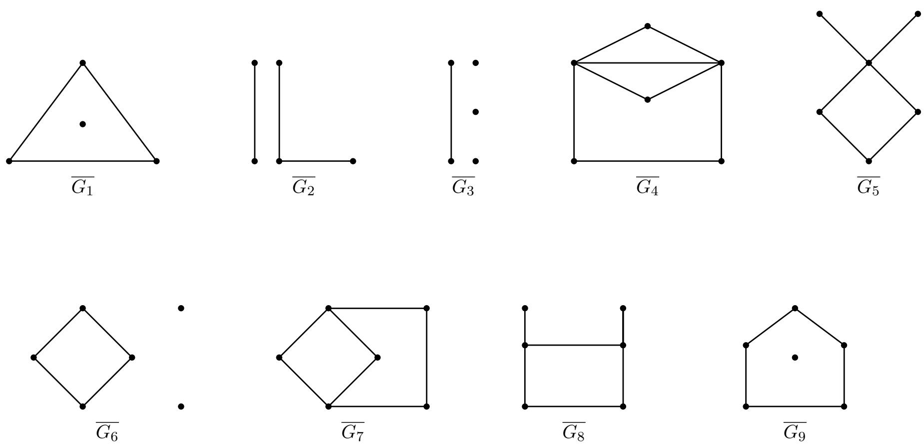

Fig. 3. Forbidden induced subgraphs of complement of line graphs

In the next theorem, we investigate when the graph $\mathbb{AG}(R)$ is the complement of a line graph.

Theorem 2.4. *Let R be a finite commutative ring which is not a field. Then $AG(R)$ is the complement of a line graph if and only if R is one of the rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ or $\mathbb{Z}_3 \times \mathbb{Z}_4$.*

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Proof. Since R is a finite commutative ring which is not a field, we can write $R \cong R_1 \times R_2 \times \cdots \times R_n$ where R_i is a local ring for all $i = 1, 2, \ldots, n$. Assume that $AG(R)$ is the complement of a line graph. Suppose that $n \geq 4$. Then the graph $AG(R)$ has an induced subgraph by the set $\{(1, 0, 0, 0, \cdots, 0), (0, 0, 1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), (1, 1, 1, 0, \cdots, 0)\}$ is isomorphic to $\overline{G}_1$. So the graph $AG(R)$ is not the complement of a line graph, a contradiction. Thus $n \leq 3$

Suppose $n = 3$ and $R \cong R_1 \times R_2 \times R_3$. Suppose that one of the rings R_i has at least 3 elements, say $|R_3| \geq 3$. Let $a \in R_3$ be an arbitrary element such that $a \notin \{0, 1\}$. It is easy to see that the induced subgraph by the set $\{(1, 0, 0), (0, 1, 1), (1, 0, a), (1, 0, 1), (1, 1, 0), (0, 0, 1)\}$ is isomorphic to $\overline{G_4}$. So the graph $\mathbb{AG}(R_1 \times R_2 \times R_3)$ is not a line graph if one of the ring R_i has at least 3 elements. Hence R_i are fields with $|R_i| = 2$ for all $1 \leq i \leq 3$. Hence the graph $\overline{AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)} \cong L(C_6)$.

Suppose $n = 2$ and $R \cong R_1 \times R_2$. At first, assume that $|Z(R_i)|^* \geq 1$ for $i = 1, 2$. Then $|U(R_i)| \geq 2$ and $ann(z_i) = Z(R_i)$ for $i = 1, 2$ and for some $z_i \in R_i^*$. Then, the induced subgraph by the set $\{(1, z_2), (z_1, z_2), (u, z_2), (1, 0)\}$ is isomorphic to $\overline{G}_1$, where u is unit in R_1. This implies that the graph $AG(R_1 \times R_2)$ is not a line graph, in this case. So, we assume that R_1 is field and R_2 is local ring but not fields. Let $|Z(R_2)^*| \geq 2$. Then there are distinct elements x and y in $Z(R_2)^*$ such that $xy = 0$ and so, the induced subgraph by the set $\{(0, x), (0, y), (0, 1), (0, u_1), (0, u_2), (1, 0)\}$, where $u_1, u_2 \in U(R_2)$ is isomorphic to $\overline{G}_3$, a contradiction. So, $|Z(R_2)| \leq 2$. If $|Z(R_2)^*| = 1$, then $R_2 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Since R_1 is field, suppose $|R_1| \geq 4$, Then, the induced subgraph by the set $\{(1, z), (v_1, z), (v_2, z), (1, 0)\}$ is isomorphic to $\overline{G}_1$, where v_1, v_2 are distinct units in R_1 other than 1 and $x \in Z(R_2)^*$. This implies that the graph $AG(R_1 \times R_2)$ is not a line graph, a contradiction. Hence $R_1 \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. If R_2 is field, then $AG(R)$ is isomorphic to complete bipartite graph. In this case, R can be one of the rings $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

It is easy to see that the graph $AG(\mathbb{F}_{q_1} \times \mathbb{F}_{q_2})$ is the complete bipartite graph $K_{q_1-1,\ q_2-1}$. Now, it is not hard to see that $AG(\mathbb{F}_{q_1} \times \mathbb{F}_{q_2})$ is the complement of the line graph of the union of two stars $K_{1,\ (q_1-1)(q_2-2)/2}$ and $K_{1,\ (q_2-1)(q_2-2)/2}$.

For rings $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, it is easy to see that $AG(R) \cong K_{2,3}$ and the complement of the line graph of the union two graphs P_3 and $K_{1,3}$. If $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ or $\mathbb{Z}_3 \times \mathbb{Z}_4$, then the complement graph of $AG(R)$ is the union of $K_4 - e$ and K_3. It is not hard to see that it is the complement of the line graph of the graph H and K_3

Finally, if $n = 1$, then R is local ring but not field and $AG(R)$ is complete. This implies that $AG(R)$ is the complement of the line graph of mK_2, where $m = |Z(R)^*|$. □

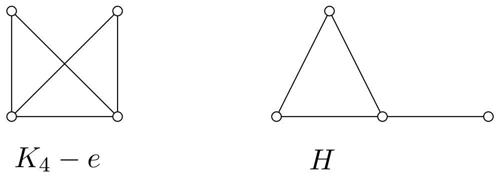

Fig. 4. $K_4 - e$ is the line graph of H

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3 The planarity and outerplanarity index of annihilator graphs

In this section, we study the planarity and outerplanarity index of annihilator graphs when R is a finite commutative ring. Also, we present all commutative ring which their annihilator graphs have planarity and outerplanarity indices.

Ghebleh et al. proposed the study of planarity and outerplanarity indexes of graphs. He gave a full characterization of graphs with respect to their planarity index.

Theorem 3.1. [13, Theorem 10] *Let G be a connected graph. Then*

(i) $\xi(G) = 0$ *if and only if G is non-planar.*

(ii) $\xi(G) = \infty$ *if and only if G is either a path, a cycle or $K_{1,3}$.*

(iii) $\xi(G) = 1$ *if and only if G is planar and either $\Delta(G) \geq 5$ or G has a vertex of degree 4 which is not a cut-vertex.*

(iv) $\xi(G) = 2$ *if and only if $L(G)$ is planar and G contains one of the graphs H_i in Fig.5 as a subgraph.*

(v) $\xi(G) = 4$ *if and only if G is one of the graphs X_k or Y_k(Fig. 5) for some $k \geq 2$.*

(vi) $\xi(G) = 3$ *otherwise.*

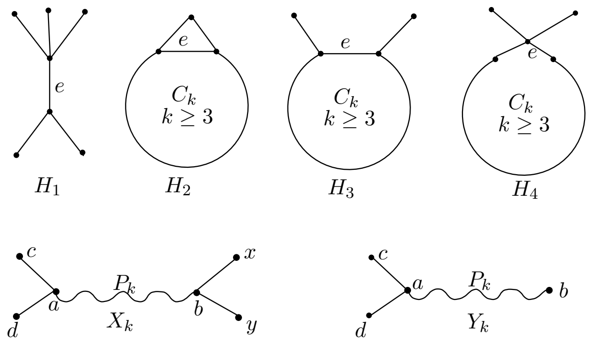

Fig. 5

In the following theorems, we determine the planar and outerplanar index of the annihilator graphs.

Theorem 3.2. *Let R be a finite commutative ring and $\mathbb{F}_q$ be a finite field with q elements. Then*

(i) $\xi(AG(R)) = \infty$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x, 2 \rangle^2}$ *or* $\frac{\mathbb{Z}_2[x,y]}{\langle x, y \rangle^2}$.

(ii) $\xi(AG(R)) = 1$ *if and only if* R *is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{F}_q$ *where* $q \geq 6$ *or* $\mathbb{Z}_3 \times \mathbb{F}_q$ *where* $q \geq 5$.

(iii) $\xi(AG(R)) = 2$ *if and only if* R *is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_{25}$ *or* $\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle}$.

(iv) $\xi(AG(R)) = 3$ *if and only if* R *is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{F}_5$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{\langle x^2 \rangle}$.

(v) $\xi(AG(R)) = 0$ *otherwise.*

Proof. Since R is a finite ring, $R = R_1 \times R_2 \times \cdots \times R_n$ for some $n \geq 1$ and each R_i is a local ring. Now, we consider the following cases:
Case 1. $n \geq 4$. In this case, as it was proved in Theorem 1.4, $AG(R)$ is non-planar. Since for every non-planar graphs we have that $\xi(G) = 0$, which implies that $\xi(AG(R)) = 0$.
Case 2. $n = 3$. In Theorem 1.4, it was proved that $AG(R)$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Fig. 2, $\Delta(AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 3$. Since the graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is planar, with using Theorem 1.2, we have $L(AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$ is planar. Also, $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ has H_4 as a subgraph. So $\xi(AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$.
Case 3. $n = 2$. By Theorem 1.4 , it was proved that $AG(R)$ is planar if and only if R is isomorphic to one of the following ring: $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$, where $\mathbb{F}$ is field.

If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, then $AG(R) \cong K_{1,\ |\mathbb{F}|-1}$. Since R is finite and $|\mathbb{F}| = q$. Hence if $q \leq 4$, then $\xi(AG(\mathbb{Z}_2 \times \mathbb{F}_q)) = \infty$. If $q = 5$, then $AG(R) \cong K_{1,4}$ and so $L(AG(\mathbb{Z}_2 \times \mathbb{F}_5)) \cong K_4$. Since $L(K_4)$ is a planar graph and H_2 as a subgraph, we have that $\xi(K_4) = 2$ which implies that $\xi(AG(\mathbb{Z}_2 \times \mathbb{F}_5)) = 3$. Clearly, if $q \geq 6$, then $\xi(AG(\mathbb{Z}_2 \times \mathbb{F}_q)) = 1$, where $q \geq 6$.

If $R \cong \mathbb{Z}_3 \times \mathbb{F}$, then $AG(R) \cong K_{2,|\mathbb{F}|-1}$. Since R is finite and $|\mathbb{F}| = q$. Hence if $q \leq 3$, then $\xi(AG(\mathbb{Z}_3 \times \mathbb{F}_q)) = \infty$. If $q = 4$, then $AG(R) \cong K_{2,3}$ is planar. As, we have the graph $L(AG(\mathbb{Z}_3 \times \mathbb{F}_4)) \cong AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is planar and $\Delta(L(AG(\mathbb{Z}_3 \times \mathbb{F}_4))) = 3$. So $L(L(AG(\mathbb{Z}_3 \times \mathbb{F}_4)))$ is planar. Also, $L(AG(\mathbb{Z}_3 \times \mathbb{F}_4))$ has H_4 as a subgraph. Thus $\xi(AG(\mathbb{Z}_3 \times \mathbb{F}_4)) = 3$. If $q = 5$, then $AG(\mathbb{Z}_3 \times \mathbb{F}_5)$ is planar and it has a vertex of degree 4 which is not a cut vertex and hence $L(AG(\mathbb{Z}_3 \times \mathbb{F}_5))$ is non planar. Clearly, if $q \geq 6$, then $\Delta(\mathbb{Z}_3 \times \mathbb{F}_q) \geq 5$ and $\xi(AG(\mathbb{Z}_3 \times \mathbb{F}_q)) = 1$, where $q \geq 6$.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{<x^2>}$, then $AG(R) \cong K_{2,3}$ is planar. Hence $L(AG(R))$ is planar and $\Delta(L(AG(R)) = 3$ by Fig. 2. So $L(L(AG(R)))$ is planar and it has H_4 as a subgraph. Thus $\xi(AG(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 3$ and $\xi(AG(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{<x^2>})) = 3$.
Case 4. If $n = 1$, then by Theorem 1.3, $AG(R)$ is planar if and only if R is isomorphic to one of the following 13 rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x,2 \rangle^2}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$, $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle}$.

If $R \cong \mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle}$, then $AG(R) \cong K_4$ and $L(K_4)$ is planar and the graphs $AG(\mathbb{Z}_{25})$, $AG(\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle})$ have H_2 as a subgraph. Hence that $\xi(AG(\mathbb{Z}_{25})) = 2$, $\xi(AG(\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle})) = 2$. Otherwise, we have $|Z(R)^*| \leq 3$. Hence $\xi(AG(R)) = \infty$. □

Theorem 3.3. [14, Theorem 3.4] *Let G be a connected graph. Then:*

(i) $\zeta(G) = 0$ *if and only if G is not outerplanar.*

(ii) $\zeta(G) = \infty$ *if and only if G is a path, a cycle or $K_{1,3}$.*

(iii) $\zeta(G) = 1$ *if and only if G is planar and G has a subgraph homeomorphic to $K_{2,3}$, $K_{1,4}$ or $K_1 + P_3$ in Fig.6.*

(iv) $\zeta(G) = 2$ *if and only if $L(G)$ is outerplanar and G has a subgraph isomorphic to one of the graphs G_2 and G_3 in Fig. 6.*

(v) $\zeta(G) = 3$ *if and only if $G \in I(d_1, d_2, \ldots, d_t)$ where $d_i \geq 2$ for $i = 2, \ldots, t-1$, and $d_1 \geq 1$ (Fig. 6).*

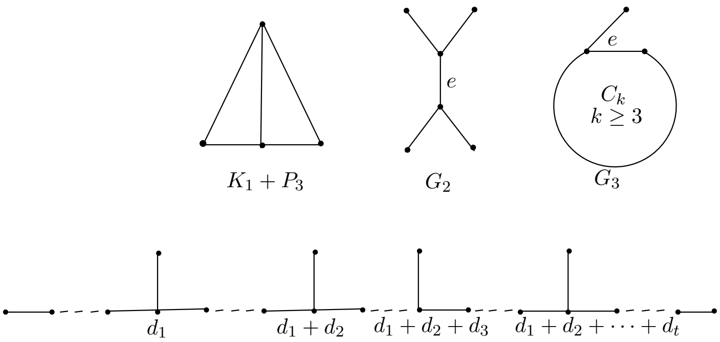

Fig.6

Now, we determine for all finite commutative rings whose outerplanarity index of their annihilator graphs.

Theorem 3.4. [18, Theorem 2.1] Let R be a finite commutative ring with identity. Then $AG(R)$ is outerplanar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$, $\mathbb{Z}_2 \times \mathbb{F}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, where $\mathbb{F}$ is field.

In the next theorems we study the outerplanar index of the annihilator graphs.

Theorem 3.5. *Let R be a finite commutative ring and $\mathbb{F}_q$ is finite field with q elements. Then*

(i) $\zeta(AG(R)) = \infty$ *if and only if R is isomorphic to one of the following ring:* $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$ *or* $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$.

(ii) $\zeta(AG(R)) = 1$ *if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$, where $q \geq 5$.*

(iii) $\zeta(AG(R)) = 0$ *otherwise*

Proof. Case 1. $n \geq 3$. In this case, as it was shown in Theorem 3.4, $AG(R)$ is non-outerplanar. Since for every non-outerplanar graphs we have that $\zeta(G) = 0$, which implies that $\zeta(AG(R)) = 0$.

Case 2. $n = 2$, In Theorem 3.4, $AG(R)$ is outerplanar if and only if $\mathbb{Z}_2 \times \mathbb{F}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, where $\mathbb{F}$ is field.

If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, then $AG(R) \cong K_{1,|\mathbb{F}|-1}$. Since R is finite and let $|\mathbb{F}| = q$. Hence if $q \leq 4$, then $\zeta(AG(\mathbb{Z}_2 \times \mathbb{F}_q)) = \infty$. If $q \geq 5$, then $AG(R) \cong K_{1,q-1}$ and so $L(AG(\mathbb{Z}_2 \times \mathbb{F}_q)) \cong K_{q-1}$ and it has K_4 as a subgraph. So K_4 is non-outerplanar and $\zeta(AG(\mathbb{Z}_2 \times \mathbb{F}_q)) = 1$ when $q \geq 5$.

If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $AG(R) \cong C_4$ and it has $\zeta(AG(\mathbb{Z}_3 \times \mathbb{Z}_3)) = \infty$.

Case 3. $n = 1$, we have $AG(R)$ is outerplanar if and only if R is isomorphic to one of the following ring: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$. Also, we know that all the ring contains at most three zero-divisor and $AG(R)$ is complete. Thus $\zeta(AG(R)) = \infty$. □

4 Generalized outerplanar of annihilator graphs

In this section, we characterize all finite commutative rings having annihilator graphs which are generalized outerplanar graphs.

In 1964, Sedláćek studied the generalized outerplanar graphs and he gave a characterization for generalized outerplanar graphs in terms of forbidden subgraphs in [17]. We state his characterization in the following theorem. A generalized outerplanar graph is a planar graph which can be embedded in the plane in such a way that at least one end-vertex of each edge lies on the external face. Moreover, for any outerplanar graph G are generalized outerplanar. Which implies that G planar graph. We state his characterization in the following theorem.

Theorem 4.1. [17] Let G be a graph. G is generalized outerplanar if and only if no subgraph of G is homeomorphic to any of the graphs in Fig. 7.

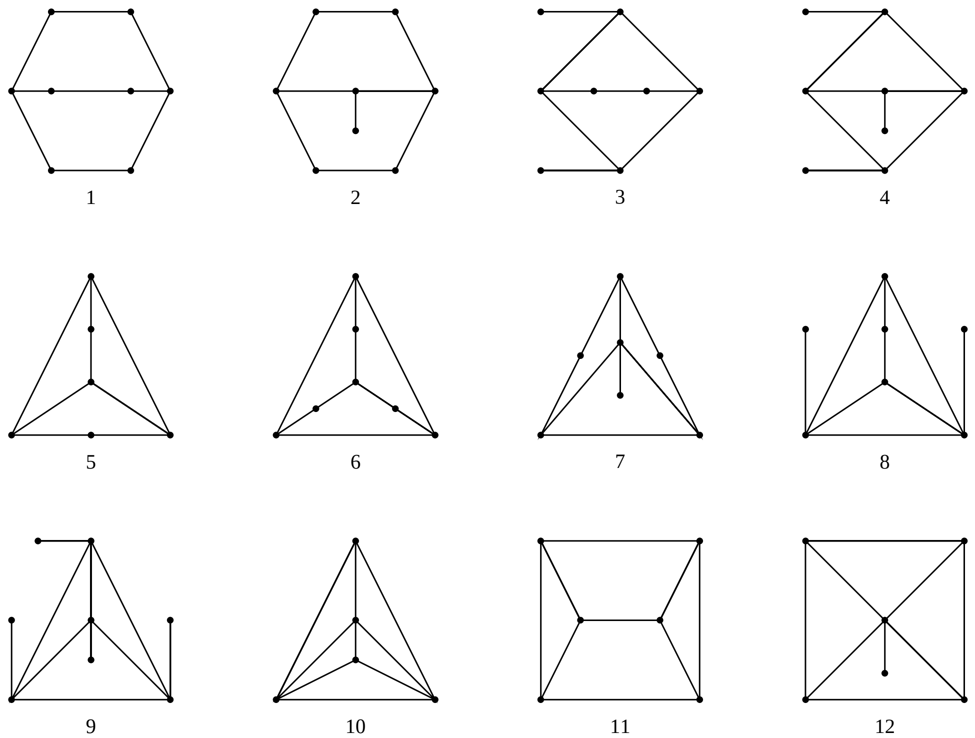

Fig.7. Forbidden subgraphs of generalized outerplanar graphs

Theorem 4.2. *Let R be a finite commutative ring with non-zero identity and $\mathbb{F}_q$ be a finite field with q elements. Then $AG(R)$ is a generalized outerplanar if and only if R is one of the following rings:* $\mathbb{Z}_2 \times \mathbb{F}_q$, $\mathbb{Z}_3 \times \mathbb{F}_q$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x,2 \rangle^2}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$, $\mathbb{Z}_{25}$ *or* $\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle}$.

Proof. We know that if $AG(R)$ is non planar, it can't be a generalized outerplanar graph. So we assume that $AG(R)$ is planar graph. Since R is finite ring, $R = R_1 \times R_2 \times \cdots \times R_n$, where $n \geq 1$ and each R_i is a local ring. Now, we consider the following cases:
Case 1. Suppose that $n \geq 4$. In [20] was proved that $AG(R)$ is non-planar. So the graph $AG(R)$ is not generalized outerplanar graph in this case.
Case 2. Assume $n = 3$. It was proved that $AG(R)$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Theorem 3.4, $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not outerplanar. Also, the graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is isomorphic to the graph of (11) of Fig. 7. Hence $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not generalized outerplanar graph.
Case 3. Assume that $n = 2$. It was proved that $AG(R)$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. It is easy to check that $AG(\mathbb{Z}_2 \times \mathbb{Z}_4) = K_{2,3}$ and $AG(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}) = K_{2,3}$. Hence the graphs $AG(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $AG(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle})$ are generalized outerplanar graphs. Let $R \cong \mathbb{Z}_2 \times \mathbb{F}$ and $R \cong \mathbb{Z}_3 \times \mathbb{F}$. It is easy to check that $AG(\mathbb{Z}_2 \times \mathbb{F}) = K_{1,|\mathbb{F}|-1}$

and $AG(\mathbb{Z}_3 \times \mathbb{F}) = K_{2,|\mathbb{F}|-1}$. Hence the graphs $AG(\mathbb{Z}_2 \times \mathbb{F})$ and $AG(\mathbb{Z}_3 \times \mathbb{F})$ are generalized outerplanar graphs.
Case 4. Assume that $n = 1$. So R is a local ring and $AG(R)$ is complete. Since $AG(R)$ is planar, by Theorem 1.3, R is isomorphic to following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2-2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x,2\rangle^2}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$, $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$. It is easy to see that $AG(R) \cong K_m$, where $m \le 4$. Therefore the graphs of all the local ring are generalized outerplanar graph. □

Now, we study the generalized outerplanar index of the graph annihilator graph. Recall that the generalized index of a graph G, denoted by $\gamma(G)$, is the smallest k such that $L^k(G)$ is not a generalized outerplanar graph and this index equals to infinity if $L^k(G)$ is a generalized outerplanar graph for all $k \ge 0$. we give a complete characterization of finite commutative rings with respect to the generalized outerplanar index of $AG(R)$ graphs. In order to do this, we use the following theorem which was proved in [10].

Theorem 4.3. [10] Let G be a connected graph. Then

(i) $\gamma(G) = 0$ if and only if G has a subgraph homeomorphic to one of the twelve graphs shown in Fig. 7.

(ii) $\gamma(G) = \infty$ if and only if G is a path, cycle or $K_{1,3}$.

(iii) $\gamma(G) = 1$ if and only if G is generalized outerplanar graph and it has a subgraph homeomorphic to one of the seven graphs shown in Fig.8.

(iv) $\gamma(G) = 2$ if and only if $L(G)$ generalized outerplanar graph and G has a subgraph from one of the five graphs shown in Fig.9.

(v) $\gamma(G) = 3$ if and only if one of the following conditions hold:

(1) $L^2(G)$ is generalized outerplanar and G has a subgraph homeomorphic to the graph of Fig.10(b).

(2) G is the graph which is drawn in Fig.10(a).

(vi) $\gamma(G) = 4$ if and only if G is one of the graphs X_k or Y_k with $k \ge 3$ Fig.5.

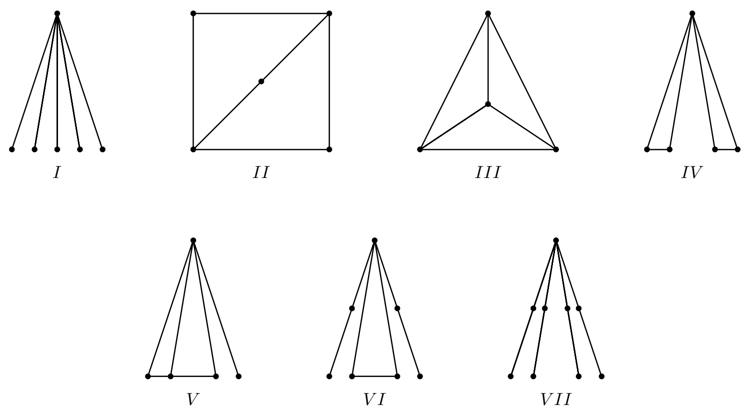

Fig.8. Forbidden subgraphs for graphs with generalized outerplanar line graphs

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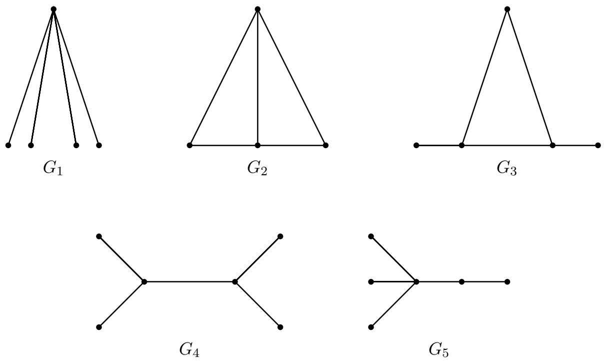
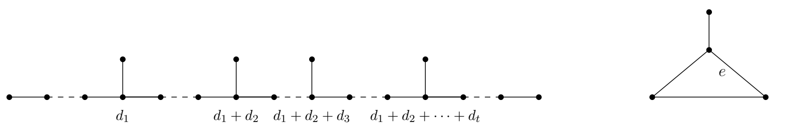


Fig.9. Forbidden subgraphs for graphs with generalized outerplanar $L^2(G)$



(a) $I(d_1, d_2, \ldots, d_t)$ where $d_i \geq 2$ for $i = 2, \ldots, t-1$, and $d_1 \geq 1$ (b)generalized outerplanar $L^3(G)$

Fig. 10

In this theorem, we characterize all of the finite commutative rings with respect to the generalized outerplanar index of their annihilator graphs.

Theorem 4.4. *Let R be a finite commutative ring and $\mathbb{F}_q$ be a finite field with q elements. Then*

1. $\gamma(AG(R)) = \infty$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x,2 \rangle^2}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$.

2. $\gamma(AG(R)) = 1$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{F}_q$ *where* $q \geq 6$, $\mathbb{Z}_3 \times \mathbb{F}_q$ *where* $q \geq 4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{\langle x^2 \rangle}$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_{25}$ *or* $\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle}$.

3. $\gamma(AG(R)) = 2$ *if and only if* $R \cong \mathbb{Z}_2 \times \mathbb{F}_5$.

4. $\gamma(AG(R)) = 0$ *otherwise*

Proof. We know that if $AG(R)$ is not a generalized outer planar graph, then $\gamma(AG(R)) = 0$. So, we may assume that $AG(R)$ is generalized outerplanar graph. Then R is one of the rings which are listed in Theorem 4.2. Clearly, by the definition of planar index and generalized outer planar index, it is easy to see that $\gamma(G) \leq \xi(G)$ for a graph G. So we have assume that $\gamma(G) \leq \xi(G)$. Now, we discuses the following cases:

Case 1. Assume that $\xi(AG(R)) = \infty$. By Theorem 3.1, $AG(R)$ is either a path, a cycle or $K_{1,3}$. Thus by Theorem 4.3, $\gamma(AG(R)) = \infty$. Now, by Theorem 3.2, R is one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x,2 \rangle^2}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$.

Case 2. Assume that $\xi(AG(R)) = 1$. So by Theorem 3.2, R is isomorphic to $\mathbb{Z}_2 \times \mathbb{F}_q$ where $q \geq 6$ or $\mathbb{Z}_3 \times \mathbb{F}_q$ where $q \geq 5$.

If $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$ where $q \geq 6$. We know $AG(R) \cong K_{1,q-1}$. So the generalized outerplanar graph which has a subgraph isomorphic to $K_{1,5}$. Therefore, by part (iii) of Theorem 4.3, we concluded that $\gamma(AG(\mathbb{Z}_2 \times \mathbb{F}_q)) = 1$ when $q \geq 6$.

If $R \cong \mathbb{Z}_3 \times \mathbb{F}_q$ where $q \geq 5$. We know $AG(R) \cong K_{2,q-1}$. So the generalized outerplanar graph and it has a subgraph isomorphic to the graph II of Fig. 8 . Therefore, by part (iii) of Theorem 4.3, we concluded that $\gamma(AG(\mathbb{Z}_3 \times \mathbb{F}_q)) = 1$ when $q \geq 5$.

Case 3. Assume that $\xi(AG(R)) = 2$. So by Theorem 3.2, R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle}$. By using Theorem 4.2, $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not generalized outer planar graph and $\gamma(AG(R)) = 0$. Also, $AG(\mathbb{Z}_{25})$ and $AG(\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle})$ are isomorphic to K_4 and these are generalized outerplanar graphs. So the graphs have a subgraph which is isomorphic to the graph III of Fig. 8. Thus $\gamma(AG(\mathbb{Z}_{25})) = 1$ and $\gamma(AG(\frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle})) = 1$.

Case 4. If $\xi(AG(R)) = 3$, then by Theorem 3.2, $R \cong \mathbb{Z}_2 \times \mathbb{F}_5$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{\langle x^2 \rangle}$. If $R \cong \mathbb{Z}_2 \times \mathbb{F}_5$, then $AG(R) \cong K_{1,4}$ and so $L(AG(\mathbb{Z}_2 \times \mathbb{F}_5)) \cong K_4$. This implies that the graph $AG(\mathbb{Z}_2 \times \mathbb{F}_5)$ and its line graph are generalized outerplanar graphs. Also, the graph $AG(\mathbb{Z}_2 \times \mathbb{F}_5)$ has subgraph which is isomorphic to the graph G_1 of Fig. 6. So, by part (iii) of Theorem 4.3, we have that $\gamma(AG(\mathbb{Z}_2 \times \mathbb{F}_5)) = 2$.

Now, let $R \cong \mathbb{Z}_3 \times \mathbb{F}_4$. Since $AG(\mathbb{Z}_3 \times \mathbb{F}_4) \cong K_{2,3}$, this graph is isomorphic to the graph II of Fig. 8. and $AG(R)$ is generalized outerplanar. Thus $\gamma(AG(\mathbb{Z}_3 \times \mathbb{F}_4)) = 1$.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{\langle x^2 \rangle}$, then $AG(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $AG(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{\langle x^2 \rangle})$ are isomorphic to $K_{2,3}$. Thus the graph is isomorphic to the graph II of Fig. 8. So $\gamma(AG(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 1$ and $\gamma(AG(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{\langle x^2 \rangle})) = 1$. □

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