Neutral Hadamard fractional integro-differential equations with impulsive and Ulam stability results in a Banach space

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Abstract

Fixed point theorems (FPTs) are fundamental tools in scientific and engineering research. This article focuses on the Ulam-Hyers stability of impulsive neutral fractional integro-differential equations (INFIDEs) involving Hadamard fractional derivatives in a Banach space. To establish the existence and uniqueness of solutions, the Banach Contraction Mapping Principle (BCMP) and Krasnoselskii's Fixed Point Theorem (KFPT) are applied.To highlight the usefulness of the theoretical insights, a carefully crafted example is introduced, enhancing and building upon prior scholarly contributions.

Keywords:Ulam Stability, Hadamard fractional integro differential equation, Fractional boundary value problems, Integro differential equations, Impulsive Conditions.

1   Introduction

Recent years have seen an increase in the study of fractional differential equations because fractional order derivatives can be used to describe memory and the inherited characteristics of a variety of phenomena. Because of this, fractional order models can depict more realistic situations than integer order models. Fractals, chaos, electrical engineering, medical science, and other fields have all made use of fractional differential equations. Fractional differential equations have witnessed significant advancement in recent years; for example, we can consult the publications [2, 3, 4, 5, 7].

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In [1],To investigate whether the Caputo-Hadamard fractional differential equation has solutions with the following impulsive boundary condition:

$$
\begin{aligned}
&{}^{CH}D_{\vartheta_k}^{\beta}x(\vartheta) = f(\vartheta, x(\vartheta)), \quad \vartheta, \vartheta_k \in [1, e], \\
&\Delta x(\vartheta_k = I_k(x(\vartheta_k))), \\
&\Delta \delta x(\vartheta_k) = \bar{I}_k(x(\vartheta_k)), \\
&x(1) = h(u), \quad x(e) = g(x),
\end{aligned}
\tag{1.1}
$$

where ${}^{CH}D^{\beta}$ is CH-FD.

In [6],We examine a neutral fractional impulsive integro-differential equation of the following form:

$$
\begin{aligned}
&{}^{C}D_{\vartheta}^{\beta}[x(\vartheta) + \int_0^{\vartheta} \frac{(\vartheta - s)^{\beta-1}}{\Gamma(\beta)} g(s, x_{\rho(s,x_s)})ds] = f(\vartheta, x_{\rho(\vartheta,x_\vartheta)}, B(x)(\vartheta)), \quad \vartheta \in [0, T], \\
&\Delta x|_{\vartheta=\vartheta_k} = I_k(x(\vartheta_k^-)), \\
&\Delta x'|_{\vartheta=\vartheta_k} = Q_k(x(\vartheta_k^-)), \\
&x(\vartheta) = \phi(\vartheta), \quad \vartheta \in [-d, 0], \\
&AX'(0) + BX'(T) = \int_0^T q(x(s))ds,
\end{aligned}
\tag{1.2}
$$

where ${}^{C}D^{\beta}$ is CFD.

The well-developed theory of impulsive differential equations of integer order has applications in mathematical modeling, particularly in the dynamics of populations that are subject to sudden changes as well as other phenomena like disease, harvesting, and so on. We direct the reader to the following sources for general theory and applications of fractional order differential equations with impulsive conditions: [8, 9, 10, 11, 12].

In [9], we investigate the theoretical analysis of solutions for a class of systems for nonlinear implicit FIDEs of Hadamard-Caputo type with fractional boundary conditions:

$$
\begin{aligned}
&{}_{H}^{C}D^{\beta}x(\vartheta) = f(\vartheta, x(\vartheta), {}_{H}^{C}D^{\beta}x(\vartheta), \int_1^{\vartheta} k(\vartheta, s, x(s))ds), \\
&x(1) = 0, \quad \gamma \, {}_{H}I^{p}x(\eta) + \zeta_{H}^{C}D^{\beta}x(\psi) = \lambda,
\end{aligned}
\tag{1.3}
$$

where the Hadamard-Caputo fractional derivative is ${}_{H}^{C}D^{\beta}$ and the conventional Hadamard fractional integral is ${}_{H}I^{p}$.

In [10], Using the topological degree approach, we investigate the existence results for neutral FIDEs with CH-derivatives:

$$
\begin{aligned}
&{}_{H}^{C}D^{\beta}[x(\vartheta) - g(\vartheta, x(\vartheta), \mathrm{P}x(\vartheta))] = f(\vartheta, x(\vartheta), \mathrm{S}x(\vartheta)), \quad \vartheta \in [0, \mathscr{L}], \\
&\mathfrak{a}x(1) + \mathfrak{b}_{H}^{C}D^{\alpha}x(1) = c_1 {}^{H}I^{\rho_1}x(\eta_1), \\
&\mathfrak{c}x(\mathscr{L}) + \mathfrak{d}_{H}^{C}D^{\alpha}x(\mathscr{L}) = c_2 {}^{H}I^{\rho_2}x(\eta_2),
\end{aligned}
$$

$$
\tag{1.4}
$$

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where $_H^C D^\beta$ is CH-FD.

Neutral functional differential equations (NFDEs) play a vital role in modeling complex systems encountered in biology, physics, and engineering, as documented in various studies [13, 14, 15, 16]. Particularly noteworthy are initial value problems (IVPs) involving fractional NFDEs with finite delays, which have gained considerable attention owing to their intrinsic ability to represent memory effects and hereditary characteristics. Numerous researchers have developed a variety of analytical techniques to investigate qualitative aspects of such equations involving classical fractional-order derivatives (FODs), as documented in [17, 22, 19].

In [12], We examine the impulsive fractional neutral pantograph integro-differential equation of the following form:

$$^C D^\beta [x(\vartheta) - \mathscr{U}(\vartheta)x(\vartheta - 1)] = f(\vartheta, u(\vartheta), x(\eta\vartheta)) + \int_0^\vartheta \mathscr{G}_1(\vartheta, s, x(s))ds, \quad \vartheta \in [0, T],$$

$$\Delta x|_{\vartheta=\vartheta_k} = I_k(x(\vartheta_k)), \tag{1.5}$$

$$x(0) = x_0,$$

$$x(\vartheta) = \phi(\vartheta), \quad \vartheta \in [-1, 0],$$

where $^C D^\beta$ is CFD.

In [19], We talk about the model's impulsive neutral integro-differential equations of fractional order:

$$^C D_\vartheta^\beta [x(\vartheta) + g(s, x_{\rho(s,x_s)})] + Ax(\vartheta) = f(s, x_{\rho(s,x_s)}, \in \vartheta_0^\vartheta e(s, x_{\rho(s,x_s)}))ds,$$

$$x_0 = g(\vartheta) \in B_h, \quad \vartheta \in (-\infty, 0], \tag{1.6}$$

$$x(\vartheta) = g_i(s, x_{\rho(s,x_s)}), \quad \vartheta \in [\vartheta_i, s_i],$$

where $^C D_\vartheta^\beta$ is CFD.

Applications for integral boundary conditions can be found in a number of practical domains, including population dynamics, thermoelasticity, chemical engineering, blood flow issues, and subterranean water flow. We direct the reader to a few recent articles [20, 21, 22] and their references for a thorough explanation of the integral boundary conditions.

Motivated by the aforementioned studies [14], we investigate the E-URs for FIDEs with IBVP of the following form:

$$^H D^\beta [x(\vartheta) - U(\vartheta, x(\vartheta))] = g(\vartheta, x(\vartheta), Bx(\vartheta)), \quad \vartheta \in \mathfrak{I} := [\mathfrak{b}, T],$$

$$\Delta x|_{\vartheta=\vartheta_k} = I_k(x(\vartheta_k^-)), \tag{1.7}$$

$$x(\mathfrak{b}) = x_0, \quad x(\mathcal{T}) = \nu \int_0^\mu x(\zeta)d\zeta, \quad \mathfrak{b} < \mu < T, \nu \in \mathbb{R},$$

where $^H D^\beta$ is the H-FD of order $1 < \beta \le 2$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function. Where $Bx(t) = \int_0^\vartheta k(\vartheta, \zeta, x(\zeta))d\zeta$ and $k : \Delta \times [\mathfrak{b}, T] \to \mathbb{R}$, $\Delta = \{(\vartheta, \zeta) : \mathfrak{b} \le \zeta \le \vartheta \le T\}$. Where $\mathfrak{b} = \vartheta_0 < \vartheta_1 < \vartheta_2 < ... < \vartheta_m = 1, \Delta x|_{\vartheta=\vartheta_k} = x(\vartheta_k^+) - x(\vartheta_k^-)$, and $x(\vartheta_k^+) = lim_{h\to 0^+} x(\vartheta_k + h)$ and $x(\vartheta_k^-) = lim_{h\to 0^-} x(\vartheta_k + h)$-the left hand and right hand limits is $x(\vartheta)$ at $\vartheta = \vartheta_k$.

Motivated by the aforementioned factors, our work aims to close the noted research gap. The following summarizes this work's main contributions.

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I) There is a void in the literature because there is little study on the examination of U-HS. This study's primary goal is to examine UHS of FIDE involving the H-FD in a Banach space.

II) The UHS is determined the existence and uniqueness of solutions for the FIDE involving the H-FD are established using Banach and KFPT.

III) To further demonstrate the practical application of the numerical findings and demonstrate their applicability and significance in resolving real-world issues, an example is also provided.

The rest of this paper is divided as follows. Section 2 describes some key lemmas and basic definitions that are the foundation of the main theoretical results. Sections 3 & 4 analyze the E-UR and Ulam stability of solutions for the problem formulations described in equations (1.7). The final section is an example to check the results.

2   Basic Results

Let $\mathscr{PC}([\mathfrak{b},T],\mathbb{R}) = \{x : [\mathfrak{b},T] \to \mathbb{R} : x \in C(\vartheta_k, \vartheta_{k+1}], \mathbb{R}\}$ be continuous. Take $x(\vartheta_k^-)$ and $x(\vartheta_k^+)$ with $x(\vartheta_k^-) = x(\vartheta_k^+)$ satisfying $\|x\|_{\mathscr{PC}} = \sup\{|x(\vartheta)| : \mathfrak{b} \le \vartheta \le T\}$.

Definition 2.1. ([14])
The Hadamard fractional integral of g is defined by

$$_H\mathfrak{I}^\beta x(\vartheta) = \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\log \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta)}{\zeta} d\zeta, \quad \beta > 0.$$

Definition 2.2. ([14])
The Hadamard fractional derivative of g is continuous function and further, $\log(\cdot) = \log_e(\cdot)$ is defined as

$$_H\mathcal{D}^\beta x(\vartheta) = \frac{1}{\Gamma(n-\beta)} \left(\vartheta \frac{d}{dt}\right)^n \int_{\mathfrak{b}}^{\vartheta} \left(\log \frac{\vartheta}{\zeta}\right)^{n-\beta-1} \frac{g(\zeta)}{\zeta} d\zeta.$$

Lemma 2.1. *Let $\beta \in \mathscr{PC}([\mathfrak{b},T],\mathbb{R})$ and $x \in \mathscr{PC}([\mathfrak{b},T],\mathbb{R})$. Then*

$$\begin{aligned}
&^H\mathcal{D}^\beta[x(\vartheta) - U(\vartheta, x(\vartheta))] = v(\vartheta), \quad \vartheta \in \mathfrak{I} := [\mathfrak{b},T], \\
&\Delta x|_{\vartheta=\vartheta_k} = I_k(x(\vartheta_k^-)), \\
&x(\mathfrak{b}) = x_0, \quad x(T) = \nu \int_0^\mu x(\zeta) d\zeta, \quad \mathfrak{b} < \mu < T, \quad \nu \in \mathbb{R},
\end{aligned} \tag{2.1}$$

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if and only if

$$
x(\vartheta) = \begin{cases}
U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\
\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right), \quad \textbf{\textit{for}} \ \vartheta \in [\mathfrak{b}, \vartheta_1), \\[4pt]
x_1 + U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\
\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right), \quad \textbf{\textit{for}} \ \vartheta \in (\vartheta_1, \vartheta_2), \\[4pt]
x_1 + x_2 + U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{t}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\
\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right), \quad \textbf{\textit{for}} \ \vartheta \in (\vartheta_2, \vartheta_3), \\[2pt]
\quad . \\
\quad . \\
\quad . \\[4pt]
U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\
\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right) \\
+ \sum_{k=1}^{m} I_k(x(\vartheta_k^-)), \quad \textbf{\textit{for}} \ \vartheta \in (\vartheta_k, \vartheta_{k+1}).
\end{cases}
$$

$$(2.2)$$

Where $N_1 = \int_{\mathfrak{b}}^{\mu} \left(\ln \frac{\zeta}{\mathfrak{b}}\right)^{\beta-1} d\zeta.$

Proof. Assume that x satisfies (2.1) and by integrating the 1st equation (2.1). If $\vartheta \in [\mathfrak{b}, \vartheta_1)$ then

$$
{}^H\mathcal{D}^\beta[x(\vartheta) - U(\vartheta, x(\vartheta))] = \gamma(\vartheta), \vartheta \in [\mathfrak{b}, T],
$$

$$
x(\mathfrak{b}) = x_0, \quad x(T) = \nu \int_0^\mu x(\zeta) d\zeta, \quad \mathfrak{b} < \mu < T, \quad \nu \in \mathbb{R}.
$$

We get

$$
x(\vartheta) = U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{t}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{t}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]}
$$

$$
\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right).
$$

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If $\vartheta \in (\vartheta_1, \vartheta_2)$ then

$$
\begin{aligned}
x(\vartheta) =& x(\vartheta_1^+) + U(\vartheta, x(\vartheta)) + x_0 - \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_1} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \\
&+ \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right)^{\beta-1} - \nu N_1 \right]} \\
&\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r} \right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right) \\
=& x(\vartheta_1^+) + x_1 + U(\vartheta, x(\vartheta)) + x_0 - \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_1} \left(\ln \frac{t}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \\
&+ \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{t} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right)^{\beta-1} - \nu N_1 \right]} \\
&\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r} \right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right) \\
=& x_1 + U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right)^{\beta-1} - \nu N_1 \right]} \\
&\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r} \right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right).
\end{aligned}
$$

If $\vartheta \in (\vartheta_2, \vartheta_3)$ then

$$
\begin{aligned}
x(\vartheta) =& x(\vartheta_2^+) + U(\vartheta, x(\vartheta)) + x_0 - \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_2} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \\
&+ \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right)^{\beta-1} - \nu N_1 \right]} \\
&\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r} \right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right)
\end{aligned}
$$

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$$=x(\vartheta_2^+) + x_2 + U(\vartheta, x(\vartheta)) + x_0 - \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_2} \left(\ln \frac{t}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta$$

$$+ \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]}$$

$$\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right)$$

$$=x_1 + x_2 + U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{t}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]}$$

$$\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right).$$

If $\vartheta \in (\vartheta_m, T)$ then

$$x(\vartheta) = U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]}$$

$$\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right)$$

$$\tag{2.3}$$

$$+ \sum_{k=1}^{m} I_k(x(\vartheta_k^-)), \texttt{ for } \vartheta \in (\vartheta_k, \vartheta_{k+1}).$$

The impulsive netural Hadamard fractional integral equation (2.2) is satifies x. $\qquad \square$

3   Main Results

To substantiate the principal findings, the following hypotheses are employed throughout the analytical framework:

(H_1): A continuous function F with some constants $K_1, K_2, N > 0$:

(i)$|F(\vartheta, \tau_1, \mathrm{v}) - F(\vartheta, \tau_2, \overline{\mathrm{v}})| \leq K_1 |\tau_1 - \tau_2| + K_2 |\mathrm{v} - \overline{\mathrm{v}}|, \; \forall \tau_1, \tau_2, \mathrm{v}, \overline{\mathrm{v}} \in \mathscr{R}, \; \vartheta \in [1, 2].$

(ii)$|z(\vartheta, \varphi, \tau_1) - z(\vartheta, \varphi, \tau_2)| \leq N |\tau_1 - \tau_2|.$

(H_2): $|U(\vartheta, \tau_1) - U(\vartheta, \tau_2)| \& \leq L_G |\tau_1 - \tau_2|$, for $\tau_1, \tau_2 \in \mathbb{R}$.

(H_3): There exists a positive constant: $K_G > 0$: $|G(\vartheta, \tau)| \leq K_G \; \forall \; \vartheta \in J$ and $\tau \in \mathscr{R}$.

(H_4): Let $K_F > 0$ be a given constant: $|F(\vartheta, \tau, \mathrm{v})| \leq K_F \; \forall \; \vartheta \in J$ and $\tau, \mathrm{v} \in \mathscr{R}$.

(H_5): Let $K_* > 0$ be a given constant: $|I_k(x(\vartheta)) - (\overline{x}(\vartheta))| \leq K^* \; \forall \; \vartheta \in J$ and $\vartheta \in \mathscr{R}$.

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Theorem 3.1. *Assume that* $(A1) - (A2)$ *is hold. If*

$$\left[L_G + \left(\frac{1}{\Gamma(\beta + 1)} \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta} + \frac{\left(\ln \frac{T}{\mathfrak{b}} \right)^{2\beta - 1}}{\Gamma(\beta + 1) \left| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta - 1} - \nu N_1 \right|} (|\nu|(\mu - \mathfrak{b}) - 1) \right) (K_1 + K_2 N) + K^* \right] < 1, \tag{3.1}$$

then the problem (1.7) *admits a unique solution on the interval* J.

Proof: Let the operator $\mathfrak{P}$ be defined as follows

$$(\mathfrak{P}x)(\vartheta) = U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta - 1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\beta - 1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right)^{\beta - 1} - \nu N_1 \right]}$$

$$\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r} \right)^{\beta - 1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta} \right)^{\beta - 1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right)$$

$$+ \sum_{k=1}^{m} I_k(x(\vartheta_k^-)).$$

At this stage, our objective is to demonstrate that the operator $\mathfrak{P}$ satisfies the contraction property.

Let $x, y \in \mathscr{PC}([\mathfrak{b}, T], \mathbb{R})$.
We have

$$|(\mathfrak{P}x)(\vartheta) - (\mathfrak{P}y)(\vartheta)| \leq |U(\vartheta, x(\vartheta)) - U(\vartheta, y(\vartheta))|$$

$$+ \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta - 1} \frac{|g(\zeta, x(\zeta), Bx(\zeta)) - g(\zeta, y(\zeta), By(\zeta))|}{\zeta} d\zeta$$

$$+ \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\beta - 1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right)^{\beta - 1} - \nu N_1 \right]}$$

$$\times (\nu \int_{\mathfrak{b}}^{\mu} \int_{r}^{\zeta} \left(\ln \frac{\zeta}{r} \right)^{\beta - 1} \frac{|g(r, x(r), Bx(r)) - g(r, y(r), By(r))|}{r} dr d\zeta$$

$$- \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta} \right)^{\beta - 1} \frac{|g(\zeta, x(\zeta), Bx(\zeta)) - g(\zeta, y(\zeta), By(\zeta))|}{\zeta} d\zeta)$$

$$+ \sum_{k=1}^{m} |I_k(x(\vartheta_k^-)) - I_k(y(\vartheta_k^-))|,$$

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$$\leq L_G \left| x(\vartheta) - y(\vartheta) \right| + \frac{1}{\Gamma(\beta+1)} \left(K_1 + K_2 N \right) \| x - y \| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta}$$

$$+ \left(\frac{\left(\ln \frac{T}{\mathfrak{b}} \right)^{2\beta-1}}{\Gamma(\beta+1) \left| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta-1} - \nu N_1 \right|} \right)$$

$$\times \left(\left(|\nu|(\mu - \mathfrak{b}) - 1 \right) \left(K_1 + K_2 N \right) \right) \| x - y \| + K^* \left| x(\vartheta) - y(\vartheta) \right|$$

$$\leq \left[L_G + \left(\frac{1}{\Gamma(\beta+1)} \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta} + \frac{\left(\ln \frac{T}{\mathfrak{b}} \right)^{2\beta-1}}{\Gamma(\beta+1) \left| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta-1} - \nu N_1 \right|} \right. \right.$$

$$\left. \left. \times \left(|\nu|(\mu - \mathfrak{b}) - 1 \right) \right) \left(K_1 + K_2 N \right) + K^* \right] \left| x(\vartheta) - y(\vartheta) \right|.$$

Thus,

$$\left| (\mathfrak{P} x_1)(\vartheta) - (\mathfrak{P} x_2)(\vartheta) \right| \leq \left[L_G + \left(\frac{1}{\Gamma(\beta+1)} \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta} + \frac{\left(\ln \frac{T}{\mathfrak{b}} \right)^{2\beta-1}}{\Gamma(\beta+1) \left| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta-1} - \nu N_1 \right|} \right. \right.$$

$$\left. \left. \times \left(|\nu|(\mu - \mathfrak{b}) - 1 \right) \right) \left(K_1 + K_2 N \right) + K^* \right] \left| x(\vartheta) - y(\vartheta) \right|.$$

Therefore, $\mathfrak{P}$ is contraction, and the expression above is strictly less than one. Consequently, the problem in (1.7) admits a unique solution on the interval $[\mathfrak{b}, T]$.

Theorem 3.2. *Under the hypothesis* $(H4)$ *&* $(H5)$, *then the problem* (1.1) *is guaranteed to possess at least one solution within in the interval* $[b, T]$.

Proof. Consider, $\mathcal{B}_r = \{ x \in \mathscr{PC}(J, \mathbb{R}) : |x| \leq r \}$.
Let Q_1 and Q_2 be two operators defined on the ball B_r, as follows:

$$(Q_1 x)(\vartheta) = U(\vartheta, x(\vartheta)) + x_0 + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right)^{\beta-1} - \nu N_1 \right]} \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r} \right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta \right.$$

$$\left. - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right) + \sum_{k=1}^{m} I_k(x(\vartheta_k^-)) \qquad (3.2)$$

and

$$(P_2 x)(\vartheta) = \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta, \qquad (3.3)$$

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Correspondingly. If $x, y \in B_{\mathrm{r}}$ then $Q_1 x + Q_2 y \in B_{\mathrm{r}}$. As we now show:

$$|Q_1 x + Q_2 y| \leq \|U(\vartheta, z(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta$$

$$+ \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta\right.$$

$$- \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right) + \sum_{i=1}^{m} x_i\|$$

$$\leq |U(\vartheta, z(\vartheta))| + |x_0| + \left|\frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{t}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right|$$

$$+ \left|\frac{\left(\ln \frac{t}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta\right.\right.$$

$$\left.\left. - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right)\right| + \left|\sum_{i=1}^{m} x_i\right|$$

$$\leq |U(\vartheta, z(\vartheta))| + |x_0| + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{t}{\zeta}\right)^{\beta-1} \left|\frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right|$$

$$+ \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\zeta}{r}\right)^{\beta-1} \left|\frac{g(r, x(r), Bx(r))}{r} dr d\zeta\right|\right.$$

$$\left. - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \left|\frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right|\right) + \left|\sum_{i=1}^{m} x_i\right|$$

$$\leq \left[L_G + \left(\frac{1}{\Gamma(\beta+1)} \left(\ln \frac{T}{\mathfrak{b}}\right)^{\beta} + \frac{\left(\ln \frac{T}{\mathfrak{b}}\right)^{2\beta-1}}{\Gamma(\beta+1) \left|\left(\ln \frac{T}{\mathfrak{b}}\right)^{\beta-1} - \nu N_1\right|}\right.\right.$$

$$\left.\left. \times \left(|\nu|(\mu - \mathfrak{b}) - 1\right)\right) (K_1 + K_2 N) + K^*\right] |x(\vartheta) - y(\vartheta)|$$

$$\leq \mathrm{r}.$$

Thus $Q_1 x + Q_2 y \in B_{\mathrm{r}}$ for all $x, y \in B_{\mathrm{r}}$.
Given that x and the operator $(Q_2 x)(\vartheta)$ are continuous, it follows that (Q_1) is a contraction

mapping. Moreover, observe that

$$|(Q_2 x)(\vartheta)| \leq \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{t}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta$$

$$\leq \frac{1}{\Gamma(\beta+1)} \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta} K_F.$$

It follows that (Q_2) is uniformly bounded throughout the domain B_r.

Let $\vartheta_1, \vartheta_2 \in J$ with $\vartheta_2 \leq \vartheta_1$, and $x \in B_r$. Since F is bounded on compact sets, $(Q_2 x)(\vartheta)$ is equicontinuous on J.

$$\sup_{(\vartheta, x, y) \in J \times B_r} |F(\varphi, x(\varphi), Bx(\vartheta))| := C_0 < \infty,$$

we will get

$$|(Q_2 x)(\vartheta_2) - (Q_2 x)(\vartheta_1)| = \left| \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_2} \left(\ln \frac{\vartheta_2}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right.$$

$$\left. - \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{t_1} \left(\ln \frac{t_1}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right|$$

$$\leq \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_2} \left(\ln \frac{\vartheta_2}{\zeta} \right)^{\beta-1} \left| \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right| d\varphi$$

$$+ \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_1} \left(\ln \frac{\vartheta_1}{\zeta} \right)^{\beta-1} \left| \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right| d\varphi$$

$$\leq \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_2} \left(\ln \frac{\vartheta_2}{\zeta} \right)^{\beta-1} C_0 \frac{d\zeta}{\zeta}$$

$$+ \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta_1} \left(\ln \frac{\vartheta_1}{\zeta} \right)^{\beta-1} C_0 \frac{d\zeta}{\zeta}$$

$$\leq \frac{C_0}{\Gamma(\beta)} [2(\vartheta_2 - \vartheta_1)^{\beta} + (\vartheta_1 - a)^{\beta} - (\vartheta_2 - a)^{\beta}]$$

$$\leq \frac{2C_0}{\Gamma(\beta)} (\vartheta_2 - \vartheta_1)^{\beta} \quad \to 0 \quad \text{as} \quad \vartheta_2 \to \vartheta_1.$$

Since $Q_2(B_r)$ is relatively compact by equicontinuity and boundedness, Q_2 is compact by the Ascoli-Arzela theorem. Hence, the problems (1.7) admit at least one fixed point on J. $\qquad\square$

4   Ulam Stability Results

The stability of a system is fundamental in both theoretical and practical contexts, ensuring reliability in applications such as transportation systems, which demand smooth and safe operations. Mathematical stability plays a critical role in the analysis of differential equations governing

such systems, which has been extensively studied by researchers, leading to valuable results (see [23, 24, 25, 26, 27]). Well known stability approach is the Ulam (sometimes Uam-Hyers) stability see e.g. ([28]). Ulam stability of FDEs has a long history and nice results (see e.g., [29, 30]). Roughly speaking, it answer the question whether an approximate solution of an equation lies close (in some sense) to its exact solution or not.

Definition 4.1. Consider the problem (1.7), and the equality is

$$\left| {}^H D^\beta x(\vartheta) - F(\vartheta, x(\vartheta), Bx(\vartheta)) \right| \leq \eta, \tag{4.1}$$

and the constant is $\mathfrak{Y}_u$ and there exist solution is

$$|x(\vartheta) - \overline{x}(\vartheta)| \leq \mathfrak{Y}_u \eta \tag{4.2}$$

Remark 4.1. *If $y \in \mathscr{PC}$ and $g_i, (i = 1, 2, \cdots, k)$ is a sequence, then y is a solution of inequality (4.1) if it satisfies its conditions.*

$(i) |x(\vartheta) \leq \eta \ (ii) {}^H D^\beta x(\vartheta) = F(\vartheta, x(\vartheta), Bx(\vartheta)) + g(\vartheta), \vartheta \in [\mathfrak{b}, T],$

Lemma 4.2. *We consider the Netural impulsive and H-FIDEs with BCs:*

$$\begin{aligned}
&{}^H D^\beta [x(\vartheta) - U(\vartheta, x(\vartheta))] = g(\vartheta, z(\vartheta), Bx(\vartheta)), \ \vartheta \in [\mathfrak{b}T], \ 0 < \delta \leq 1, \\
&\Delta x|_{\vartheta=\vartheta_k} = I_k(x(\vartheta_k^-)), \\
&x(\mathfrak{b}) = x_0, \quad x(T) = \nu \int_0^\mu x(\zeta)d\zeta, \quad \mathfrak{b} < \mu < T, \nu \in \mathbb{R},,
\end{aligned} \tag{4.3}$$

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then

$$x(\vartheta) = \begin{cases} U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{t}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\ \quad \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right), \; \boldsymbol{for} \; \vartheta \in [\mathfrak{b}, \vartheta_1), \\[2mm] x_1 + U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{t}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\ \quad \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right), \; \boldsymbol{for} \; \vartheta \in (\vartheta_1, \vartheta_2), \\[2mm] x_1 + x_2 + U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\ \quad \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right), \; \boldsymbol{for} \; \vartheta \in (\vartheta_2, \vartheta_3), \\ \quad . \\ \quad . \\ \quad . \\[2mm] U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta + \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}}\right)^{\beta-1}}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right)^{\beta-1} - \nu N_1\right]} \\ \quad \times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{\mathfrak{b}}^{\zeta} \left(\ln \frac{\varsigma}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right) \\ \quad + \sum_{k=1}^{m} I_k(x(\vartheta_k^-)), \; \boldsymbol{for} \; \vartheta \in (\vartheta_k, \vartheta_{k+1}). \end{cases}$$

$$(4.4)$$

Where $N_1 = \int_{\mathfrak{b}}^{\mu} \left(\ln \frac{\zeta}{\mathfrak{b}}\right)^{\beta-1} d\zeta$. Moreover, we get

$$\left| x(t) - U(\vartheta, x(\vartheta)) - x_0 - \frac{1}{\Gamma(\beta)} \int_{\mathfrak{b}}^{\vartheta} \left(\ln \frac{\vartheta}{\zeta}\right)^{\beta-1} \frac{g(t, x(\zeta), Bx(\zeta))}{\zeta} d\zeta - \frac{\left(\ln \frac{t}{\mathfrak{b}}\right)}{\Gamma(\beta)\left[\left(\ln \frac{T}{\zeta}\right) - \nu N_1\right]} \right.$$

$$\times \left(\nu \int_{\mathfrak{b}}^{\mu} \int_{r}^{\zeta} \left(\ln \frac{\varsigma}{r}\right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta \right.$$

$$\left. \left. - \int_{\mathfrak{b}}^{T} \left(\ln \frac{T}{\zeta}\right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta\right) + \sum_{k=1}^{m} I_k(x(\vartheta_k^-)) \right|,$$

$$\leq \left[\frac{\left(\ln \frac{T}{\mathfrak{b}}\right)^{2\beta-1}}{\Gamma(\beta+1) \left| \left(\ln \frac{T}{\mathfrak{b}}\right)^{\beta-1} - \nu N_1 \right|} \times (|\nu|(\mu - \mathfrak{b}) - 1)\epsilon + K^* \right] \eta.$$

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Theorem 4.3. *The Hypothesis $(H_1 - H_4)$ hold.*

$$
{}^H D^\beta[x(\vartheta) - U(\vartheta, x(\vartheta))] = F(\vartheta, x(\vartheta), Bx(\vartheta)), \ \vartheta \in [\mathfrak{b}, T], \ 0 < \beta \le 1,
$$
$$
\Delta x|_{\vartheta = \vartheta_k} = I_k(x(\vartheta_k^-)),
$$
$$
x(\mathfrak{b}) = x_0, \quad x(T) = \nu \int_0^\mu x(\zeta) d\zeta, \quad \mathfrak{b} < \mu < T, \nu \in \mathbb{R}.
$$

(4.5)

Then the problem (1.1) *is Hyers-Ulam, under lemma (4.1), if*

$$
\Omega : \left[L_G + \left(\frac{1}{\Gamma(\beta+1)} \left(\ln \frac{T}{\mathfrak{b}} \right)^\beta + \frac{\left(\ln \frac{T}{\mathfrak{b}} \right)^{2\beta-1}}{\Gamma(\beta+1) \left| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta-1} - \nu N_1 \right|} \right. \right.
$$
$$
\times \left. \left. (|\nu|(\mu - \mathfrak{b}) - 1) \right) (K_1 + K_2 N) + K^* \right] \le 1
$$

(4.6)

Proof. Let $x \in \mathscr{PC}(J, \mathbb{R})$ be a solution of (4.1), then

$$
|y(\vartheta) - x(\vartheta)| \le \left| y(\vartheta) - \left(U(\vartheta, x(\vartheta)) + x_0 + \frac{1}{\Gamma(\beta)} \int_\mathfrak{b}^\vartheta \left(\ln \frac{t}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right. \right.
$$
$$
+ \frac{\left(\ln \frac{\vartheta}{b} \right)}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right) - \nu N_1 \right]} \times \left(\nu \int_\mathfrak{b}^\mu \int_r^\zeta \left(\ln \frac{\zeta}{r} \right)^{\beta-1} \frac{g(r, x(r), Bx(r))}{r} dr d\zeta \right.
$$
$$
\left. \left. - \int_\mathfrak{b}^T \left(\ln \frac{T}{\zeta} \right)^{\beta-1} \frac{g(\zeta, x(\zeta), Bx(\zeta))}{\zeta} d\zeta \right) + \sum_{k=1}^m I_k(x(\vartheta_k^-)) \right) \right|
$$
$$
+ \left(|U(\vartheta, y(\vartheta)) - U(\vartheta, x(\vartheta))| + \frac{1}{\Gamma(\beta)} \int_\mathfrak{b}^\vartheta \left(\ln \frac{\vartheta}{\zeta} \right)^{\beta-1} \right.
$$
$$
|g(\zeta, y(\zeta), By(\zeta)) - g(\zeta, x(\zeta), Bx(\zeta))| \frac{d\zeta}{\zeta} + \frac{\left(\ln \frac{\vartheta}{b} \right)}{\Gamma(\beta) \left[\left(\ln \frac{T}{\zeta} \right) - \nu N_1 \right]}
$$
$$
\times \left(\nu \int_\mathfrak{b}^\mu \int_r^\zeta \left(\ln \frac{\zeta}{r} \right)^{\beta-1} |g(r, y(r), By(r)) - g(r, x(r), Bx(r))| \frac{dr}{r} d\zeta \right.
$$
$$
\left. - \int_\mathfrak{b}^T \left(\ln \frac{T}{\zeta} \right)^{\beta-1} |g(\zeta, y(\zeta), By(\zeta)) - g(\zeta, x(\zeta), Bx(\zeta))| \frac{d\zeta}{\zeta} \right)
$$
$$
\left. + \sum_{k=1}^m |I_k(y(\vartheta_k^-)) - I_k(y(\vartheta_k^-))| \right)
$$
$$
\le \gamma_1 \eta + \left[L_G + \left(\frac{1}{\Gamma(\beta+1)} \left(\ln \frac{T}{\mathfrak{b}} \right)^\beta + \frac{\left(\ln \frac{T}{\mathfrak{b}} \right)^{2\beta-1}}{\Gamma(\beta+1) \left| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta-1} - \nu N_1 \right|} \right. \right.
$$
$$
\times \left. \left. (|\nu|(\mu - \mathfrak{b}) - 1) \right) (K_1 + K_2 N) + K^* \right] |y - x|_{PC}.
$$

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where

$$\gamma_1 = \frac{\left(\ln \frac{T}{\mathfrak{b}}\right)^{2\beta-1}}{\Gamma(\beta+1)\left|\left(\ln \frac{T}{\mathfrak{b}}\right)^{\beta-1} - \nu N_1\right|} \times (|\nu|(\mu-\mathfrak{b})-1)\epsilon + K^*$$

$\Rightarrow,$

$$|y(\vartheta) - x(\vartheta)| \le \gamma_1\eta + \left[L_G + \left(\frac{1}{\Gamma(\beta+1)}\left(\ln \frac{T}{\mathfrak{b}}\right)^{\beta} + \frac{\left(\ln \frac{T}{\mathfrak{b}}\right)^{2\beta-1}}{\Gamma(\beta+1)\left|\left(\ln \frac{T}{\mathfrak{b}}\right)^{\beta-1} - \nu N_1\right|}\right.\right.$$
$$\left.\left. \times (|\nu|(\mu-\mathfrak{b})-1)\right)(K_1 + K_2 N) + K^*\right]|y-x|_{\mathscr{PC}}. \tag{4.7}$$

Hence, from (4.7), we get

$$|y(\vartheta) - x(\vartheta)| \le \frac{\gamma_1}{1-\Omega}\eta. \tag{4.8}$$

Hence, the solution is (1.1) is Hyers-Ulam stable . $\square$

5   Example

Example 5.1. *We examine the Hadamard fractional problem as follows:*

$$^H D^{4/3}x(\vartheta) - \frac{tan^{-1}|x(\vartheta)|}{35} = \frac{\vartheta^3 + sin|x(\vartheta)|}{45} + \int_0^\vartheta \frac{1}{50}(\vartheta^2 + \varphi^2)\vartheta(\varphi)d\varphi, \tag{5.1}$$

$$\Delta x(\vartheta) = \frac{\vartheta(1^-/2)}{3 + \vartheta(1^-/2)}, \tag{5.2}$$

$$x(b) = x_0, \ x(T) = \nu \int_1^2 \frac{|x(\kappa)|}{10 + |x(\kappa)|}d\kappa. \tag{5.3}$$

Set

$$U(\vartheta, x(\vartheta)) = \frac{tan^{-1}|z(\vartheta)|}{35},$$
$$F(\vartheta, x(\vartheta), Bx(\vartheta)) = \frac{\vartheta^3 + sin|x(\vartheta)|}{45}$$
$$Bx(\vartheta)) = \int_0^\vartheta \frac{1}{50}(\vartheta^2 + \varphi^2)\vartheta(\varphi)d\varphi$$

and

$$g(t, x(\kappa)) = \int_1^2 \frac{|x(\kappa)|}{10 + |x(\kappa)|}.$$

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Hence the assumptions $(H_1 - H_4)$ holds. Where $1 < \beta \le 2$, $b = 1$, $T = 2$, $K_1 = K_2 = \frac{1}{45}$, $N = \frac{1}{750}$, $L_G = \frac{1}{35}$, $K_G = 110$, $\delta = \frac{1}{2}, K^* = \frac{1}{3}$, $\nu = 0.5, N_1 = 0.053$ By using Theorem (3.1), we conclude that:

$$\left[L_G + \left(\frac{1}{\Gamma(\beta+1)} \left(\ln \frac{T}{\mathfrak{b}} \right)^\beta + \frac{\left(\ln \frac{T}{\mathfrak{b}} \right)^{2\beta-1}}{\Gamma(\beta+1) \left| \left(\ln \frac{T}{\mathfrak{b}} \right)^{\beta-1} - \nu N_1 \right|} (|\nu|(\mu - \mathfrak{b}) - 1) \right) (K_1 + K_2 N) + K^* \right] < 1.$$

According to Theorem (3.1), there is a unique solution to the problem $(5.1) - (5.3)$ on $[1, 2]$. Further, we get the solution x of the problem, $(5.1) - (5.3)$ is below

$$x(\vartheta) = 1 + \frac{1}{\Gamma(\frac{4}{3})} \int_1^2 \left(\ln \frac{\vartheta}{\zeta} \right)^{\frac{1}{3}} \frac{\mathfrak{e}^{-\vartheta} |x|}{12 + \mathfrak{e}^\vartheta (1 + |x|)} + \frac{1}{2} \int_0^\vartheta \mathfrak{e}^{-(\kappa-\vartheta)} x(\kappa) \frac{d\kappa}{\kappa}$$

$$+ \frac{\left(\ln \frac{\vartheta}{\mathfrak{b}} \right)^{\frac{1}{3}}}{\Gamma(\frac{4}{3}) \left[\left(\ln \frac{T}{\zeta} \right)^{\frac{1}{3}} - \nu N_1 \right]}$$

$$\times \left(\nu \int_{\mathfrak{b}}^\mu \int_r^\zeta \left(\ln \frac{\zeta}{r} \right)^{\frac{1}{3}} \frac{1}{\Gamma(\frac{4}{3})} \int_1^2 \left(\ln \frac{\vartheta}{\zeta} \right)^{\frac{1}{3}} \frac{\mathfrak{e}^{-\vartheta} |x|}{12 + \mathfrak{e}^t (1 + |x|)} + \frac{1}{2} \int_0^\vartheta \mathfrak{e}^{-(\kappa-t)} z(\kappa) \frac{d\kappa}{\kappa} \right.$$

$$\left. - \int_{\mathfrak{b}}^T \left(\ln \frac{T}{\zeta} \right)^{\frac{1}{3}} \frac{1}{\Gamma(\frac{4}{3})} \int_1^2 \left(\ln \frac{t}{\zeta} \right)^{\frac{1}{3}} \frac{\mathfrak{e}^{-\vartheta} |x|}{12 + \mathfrak{e}^\vartheta (1 + |x|)} + \frac{1}{2} \int_0^\vartheta \mathfrak{e}^{-(\kappa-\vartheta)} x(\kappa) \frac{d\kappa}{\kappa} \right)$$

We obtain

$$|\mathrm{y} - x| = \frac{\gamma_1}{1 - \Omega} \eta \le 1.0784.$$

This establishes that the system of equations $(5.1) - (5.3)$ exhibits Ulam-Hyers stability with respect to the norm $\upsilon_r = 1$.

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